

CS103, AUTUMN 2026



Lecture 00:

Intro to Set Theory

Are there “laws of physics”
in computer science?

For this, along with further course motivation and Day 1 logistics, see the Prezi included on today's lecture page.

Key Questions in CS103

- ***Computability Theory***
 - What problems can you solve with a computer?
- ***Complexity Theory***
 - Why are some problems harder to solve than others?
- ***Discrete Mathematics***
 - How can we be certain in our answers to these questions?

Course Website

[*https://cs103.stanford.edu*](https://cs103.stanford.edu)

All course content
will be hosted
here, except for
lecture videos.

Problem Set 0

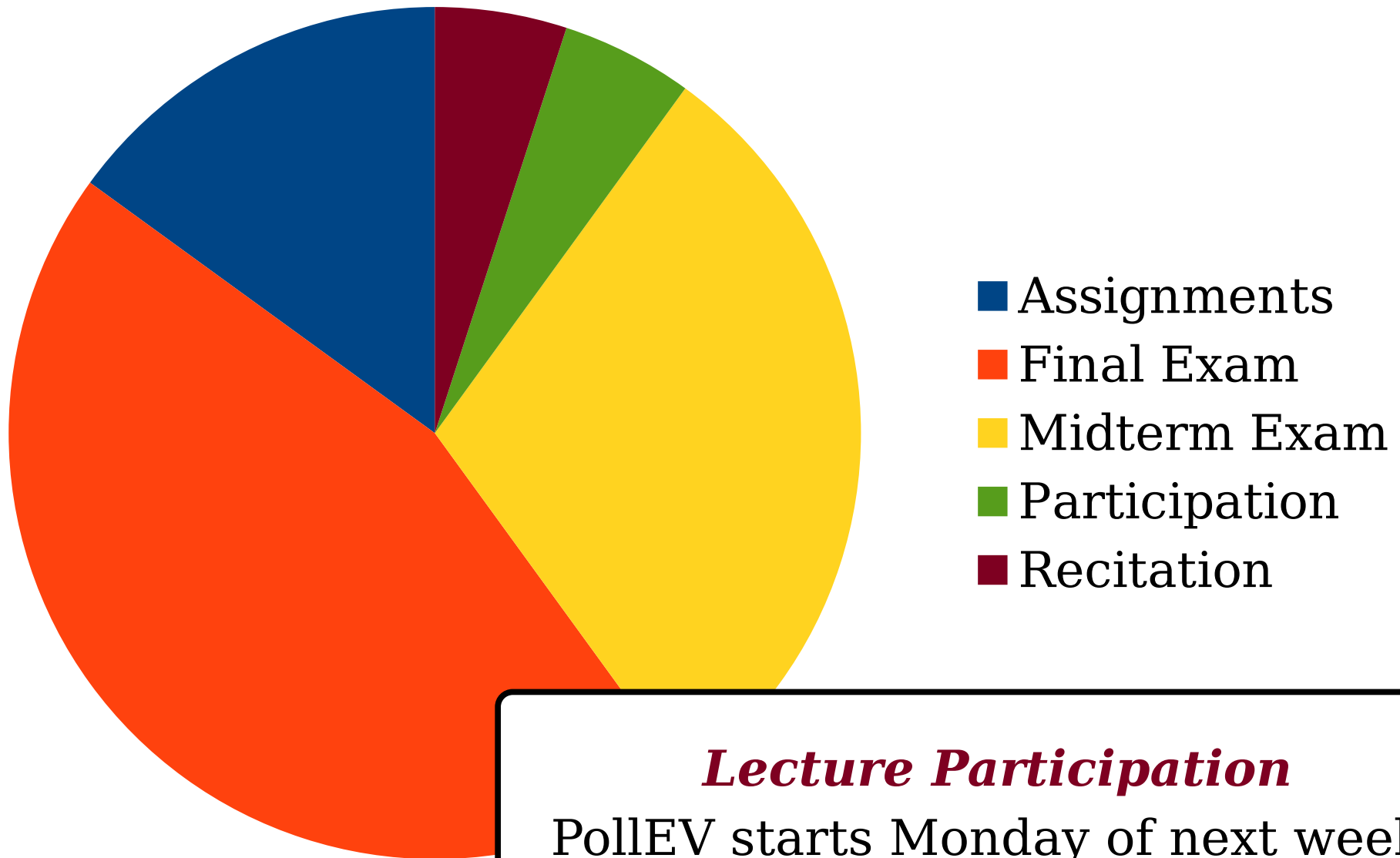
- Your first assignment, Problem Set 0, goes out today. It's due Friday at 11:59 PM Pacific.
- This assignment requires you to set up your development environment and to get set up on Gradescope and EdStem.
- There's no coding involved, but it's good to start early in case you encounter any technical issues.



Things Due This Friday

- These are all due this Friday at 11:59 PM:
 - ***Problem Set 0***
 - ***Problem Set Matchmaker Form***
 - ***Recitation Preference Form***
- We are working behind the scenes to finalize recitation options. Stay tuned! We will post links for that and the partner matchmaker form on Ed as they become available.

Grading



Approaching this Course

- Key recommendations:
 - ***Attend lecture in person.*** This ensures you stay current with the concepts and gives time to digest the content.
 - ***Take notes by hand.*** Explaining concepts in your own words improves learning and identifies questions to ask.
 - ***Ask questions!*** That's what we're here for. Feel free to ask in lecture, after class, on EdStem, or in office hours.
 - ***Read the Guides.*** We post readings on the website in the form of "Guide to X." Those are the main readings for the course.
 - ***Focus on learning.*** Problem sets are worth much less than the exams. Prioritize building skills over completion. Avoid shortcuts.
- CS103 is more like building a rocket than learning a language: there's less frequent feedback you'll need to review in more depth.

***Let's get started on this
incredible journey!***

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

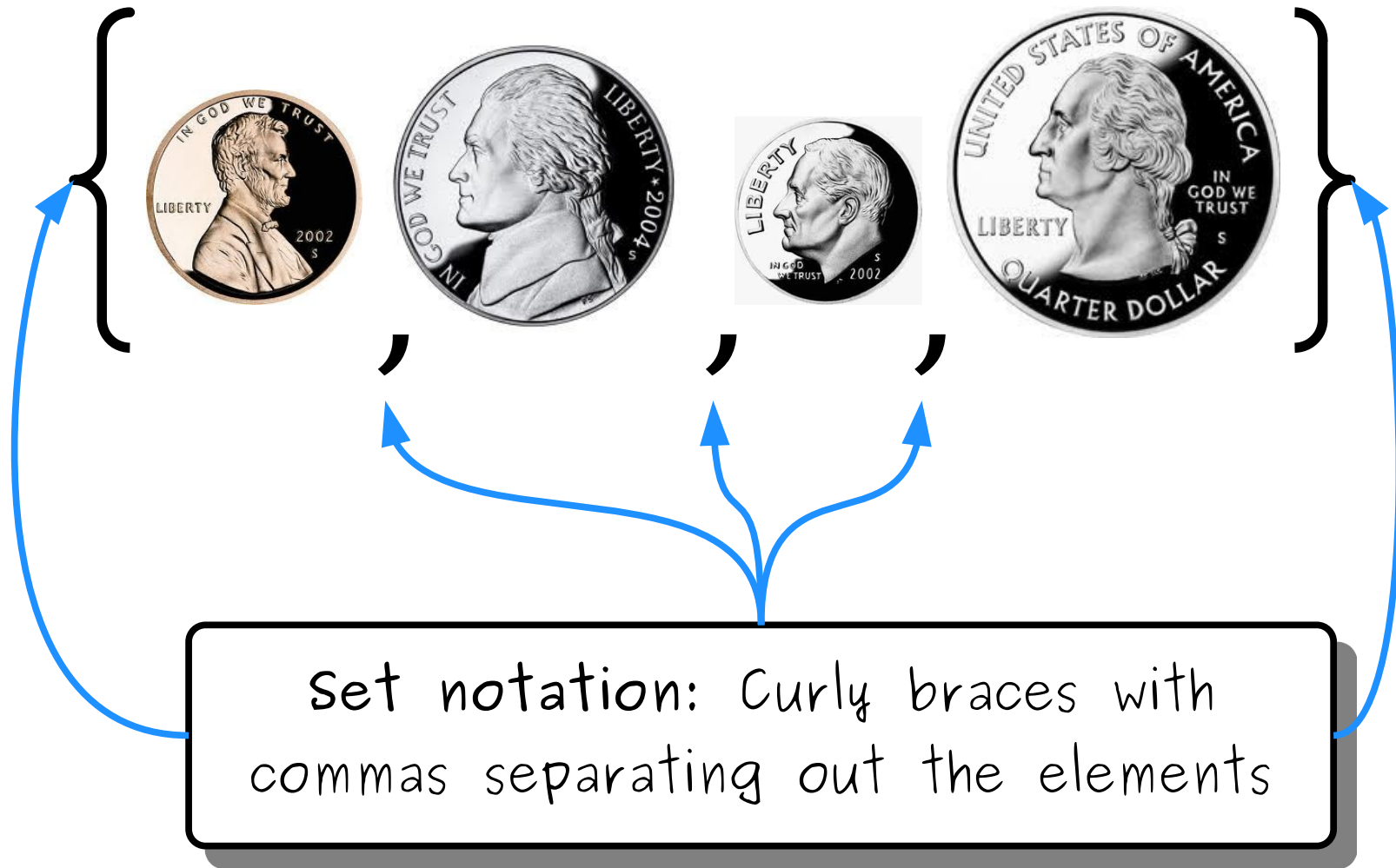
Introduction to Set Theory

- 1. Sets and Elements (Definitions and Notation)**
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

A ***set*** is an unordered collection of distinct objects,
which may be anything, including other sets.



A **set** is an unordered collection of distinct objects,
which may be anything, including other sets.



A **set** is an unordered collection of distinct objects, which may be anything, including other sets.

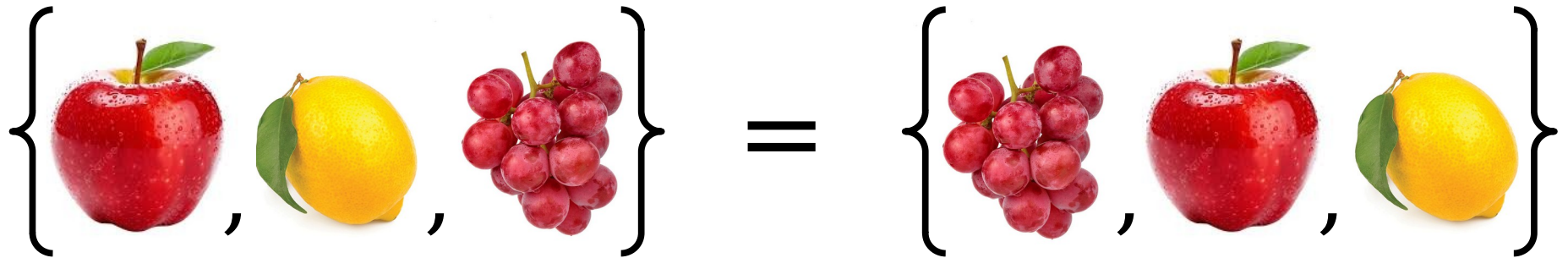
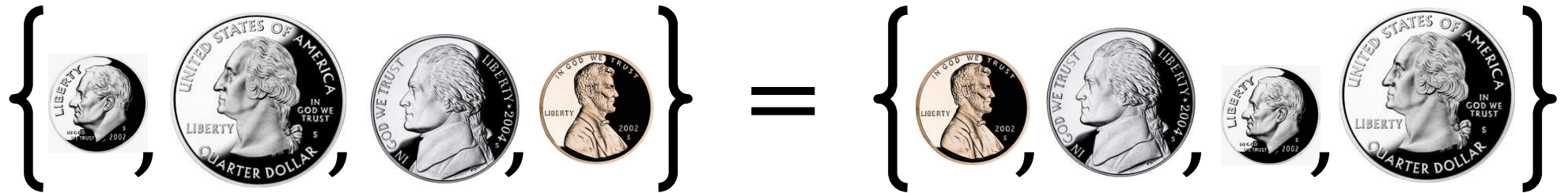
Two sets are equal when they have the same contents, ignoring order.



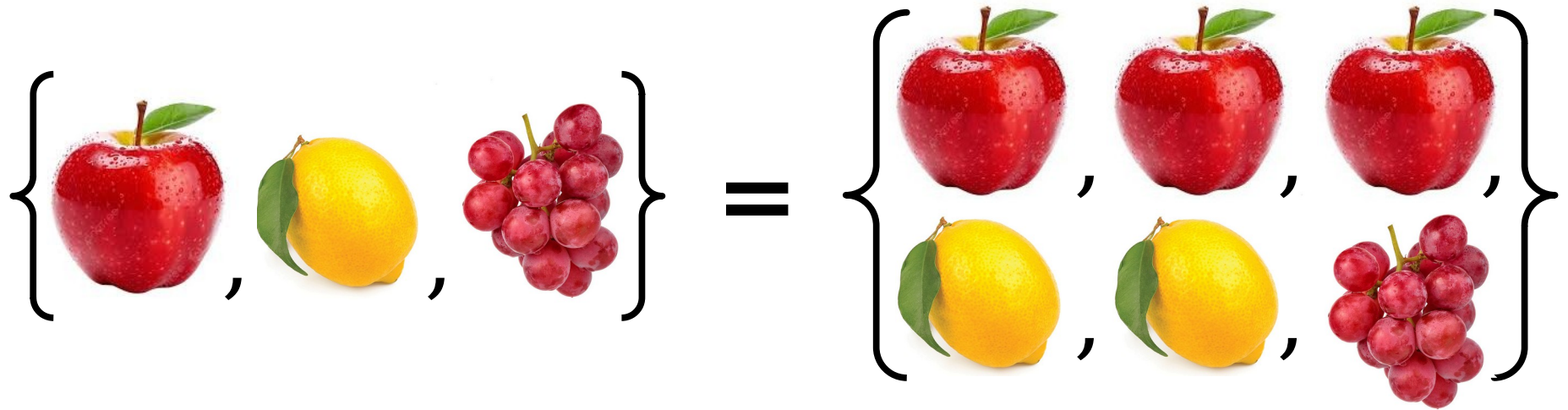
Two sets are equal when they have the same contents, ignoring order.



Two sets are equal when they have the same contents, ignoring order.



Two sets are equal when they have the same contents, ignoring order.



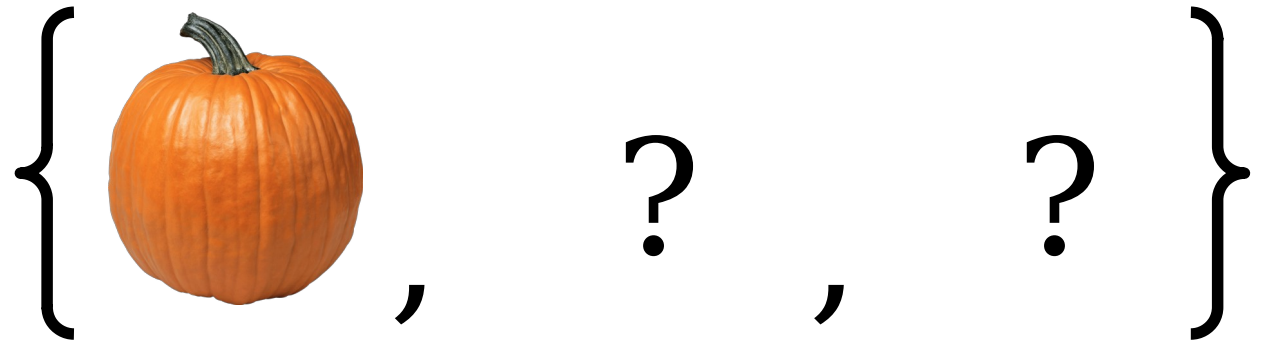
These are two descriptions of the same set. On the right, the exact same apple is listed multiple times; same for the lemon.

Sets cannot contain duplicate elements.
Any repeated elements are ignored.

The objects that make up a set are called the ***elements*** of that set.

$\{ \quad ? \quad , \quad ? \quad , \quad ? \quad \}$

The objects that make up a set are called the ***elements*** of that set.



The objects that make up a set are called the ***elements*** of that set.



The objects that make up a set are called the ***elements*** of that set.



The objects that make up a set are called the ***elements*** of that set.



This symbol means "is
an element of."

The objects that make up a set are
called the ***elements*** of that set.



€



This symbol means "is not an element of."

The objects that make up a set are called the **elements** of that set.



The doughnut is an element of the set.



Question: Is this an element of the set?

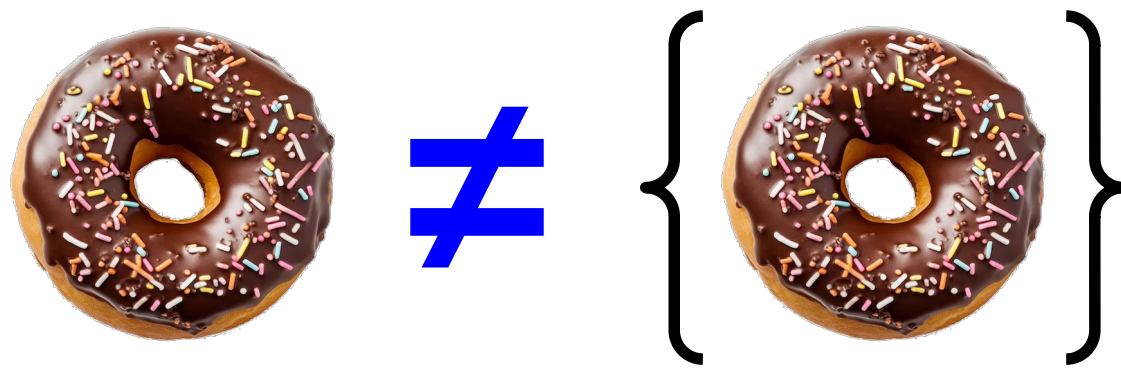


Question: Is this an element of the set?

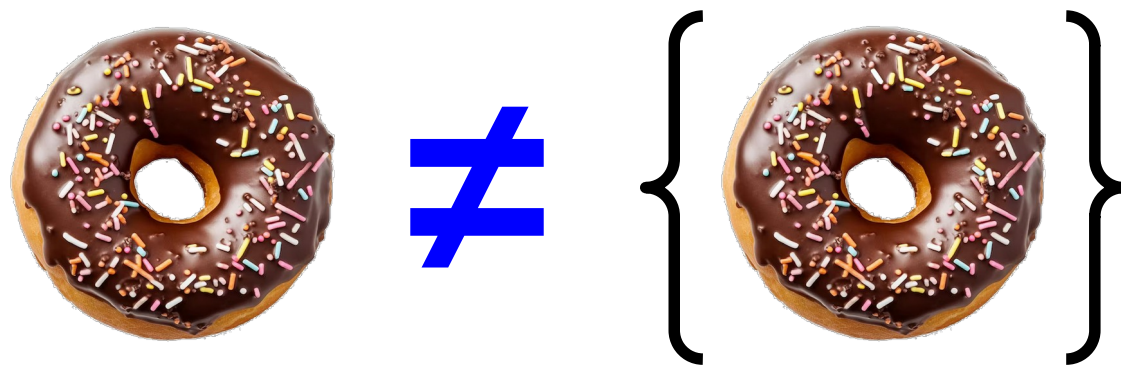


Important distinction: $\left\{ \text{donut} \right\} \neq \text{donut}$

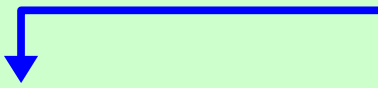
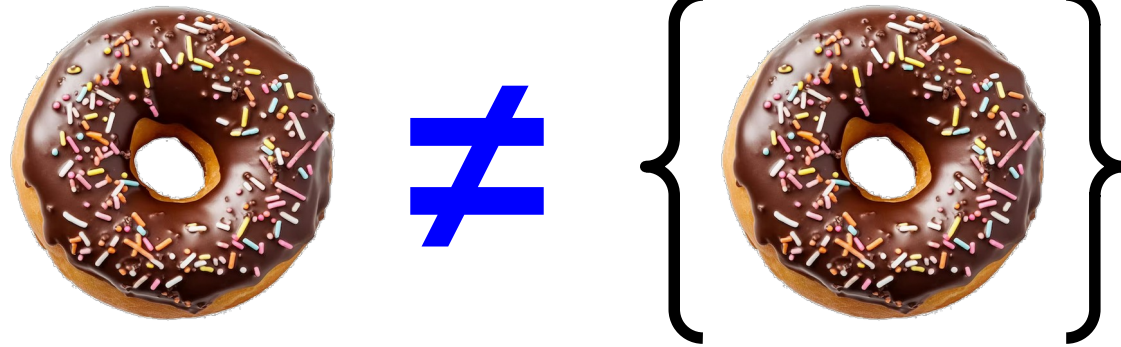
Question: Is this an element of the set?



No object x is equal to the set containing x .



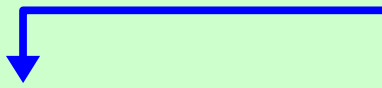
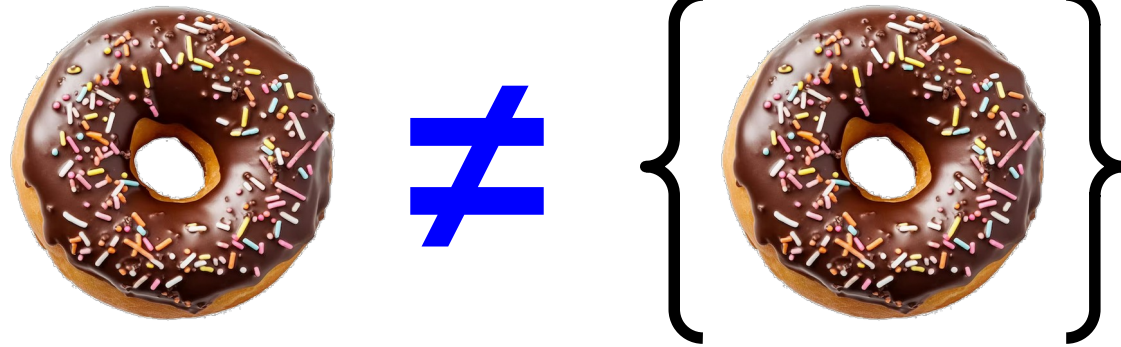
No object x is equal to the set containing x .



This is a
doughnut.



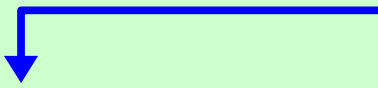
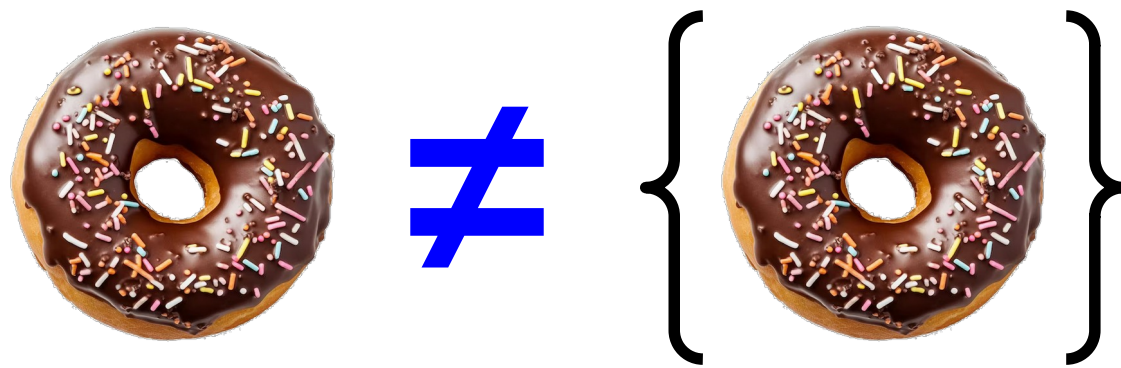
No object x is equal to the set containing x .



This is a
doughnut.



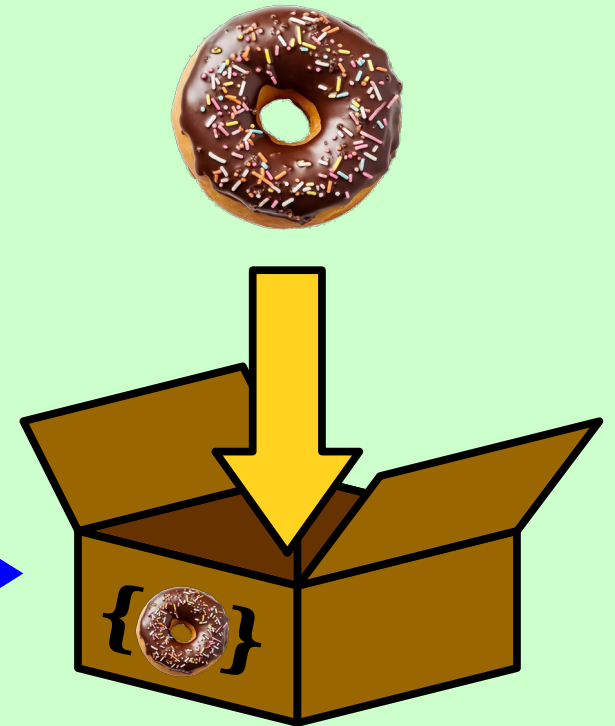
No object x is equal to the set containing x .



This is a
doughnut.

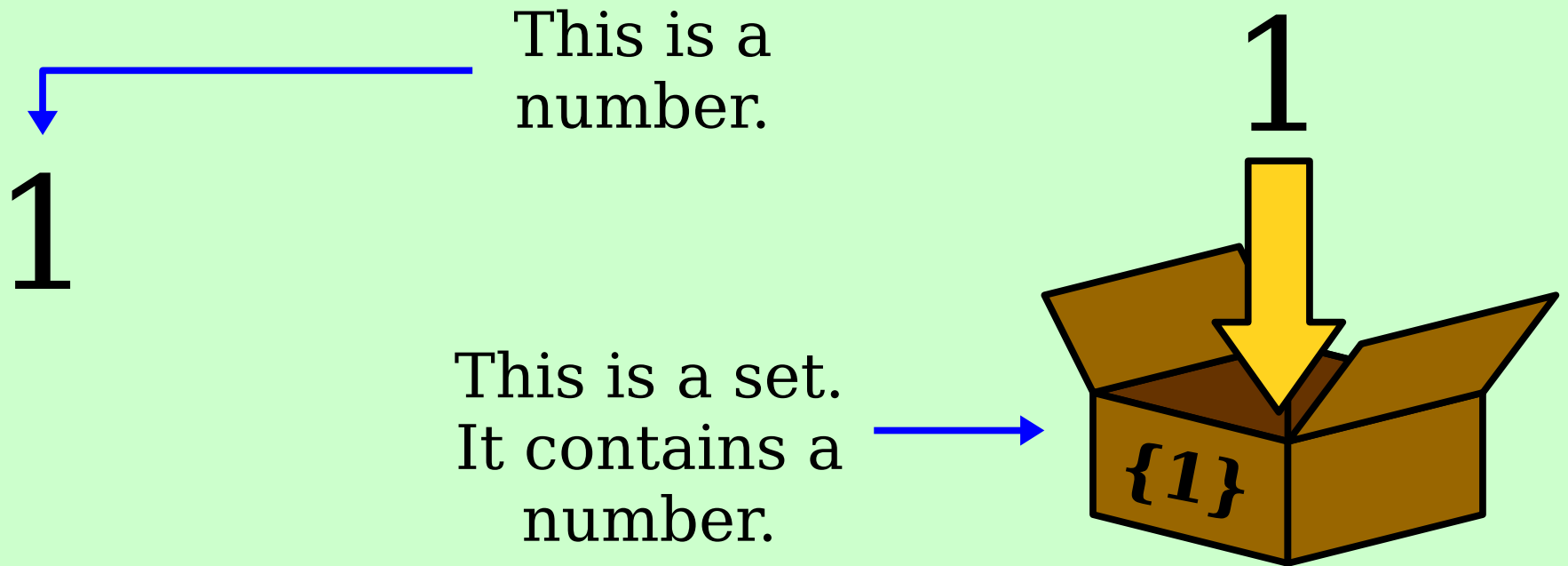


This is a set.
It contains a
doughnut.



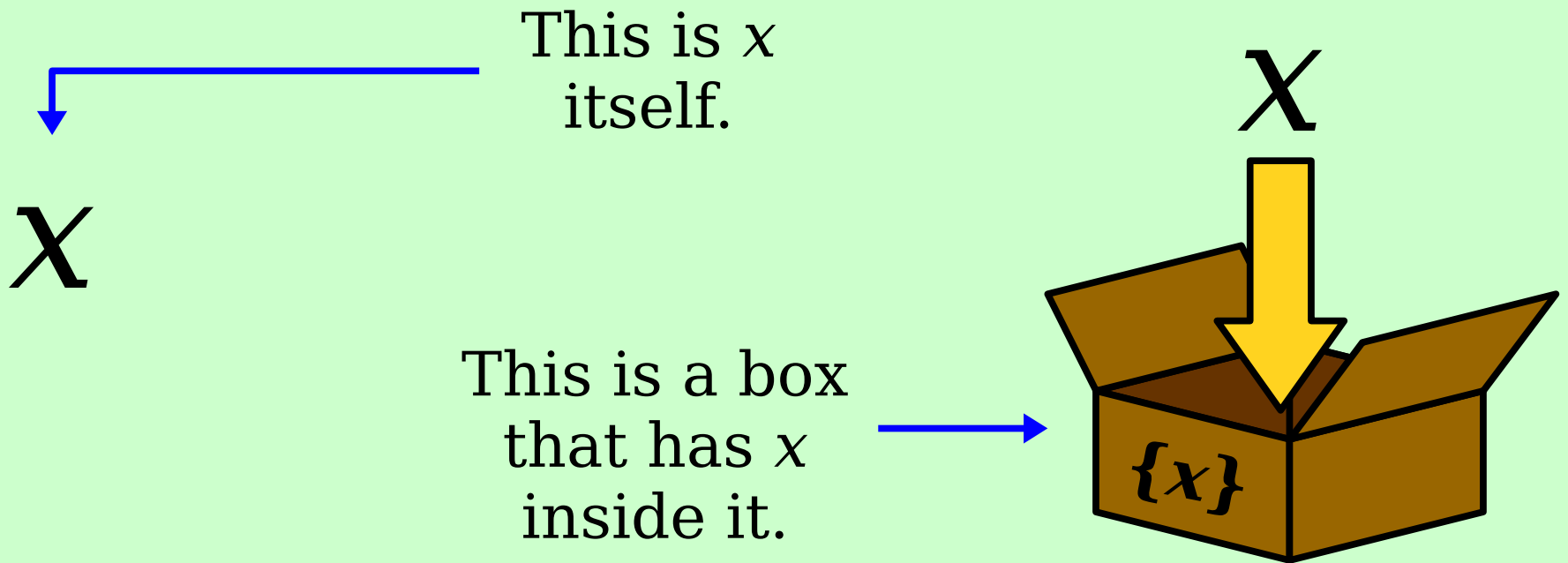
No object x is equal to the set containing x .

$$1 \neq \{1\}$$



No object x is equal to the set containing x .

$$x \neq \{x\}$$



No object x is equal to the set containing x .

Introduction to Set Theory

- 1. Sets and Elements (Definitions and Notation)**
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)

2. Subsets

3. The Empty Set

4. Set Builder Notation and Some Important Sets

5. Set Operations

6. Power Sets

7. Cardinality

8. Cantor's Theorem

9. Shocking Implications

Subsets

- A set S is called a **subset** of a set T (denoted $S \subseteq T$) when all elements of S are also elements of T .
- Examples:
 - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
 - $\{ c, b \} \subseteq \{ a, b, c, d \}$
 - $\{ \text{🍏}, \text{🍋}, \text{🍇} \} \subseteq \{ \text{🍏}, \text{🍋}, \text{🍇} \}$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\text{pumpkin} \stackrel{?}{\in} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

 $\in S$

An upward-pointing arrow connects the pumpkin in the membership statement to the pumpkin element within the set S.

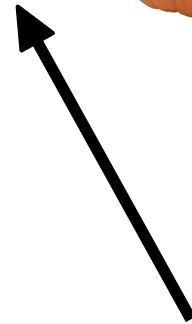
Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$2 \overset{?}{\in} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$



$$2 \in S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\{2\} \overset{?}{\in} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\{2\} \in S$$

Elements and Subsets

$$S = \{ 2, \text{🍂}, \{2\}, \text{🍩}, \text{👤} \}$$

$$\{2, \text{👤}\} \overset{?}{\in} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\{2, \text{Brad Pitt}\} \notin S$$

General intuition:
 $x \in S$ means you
can ***point at x***
inside of S .

Elements and Subsets

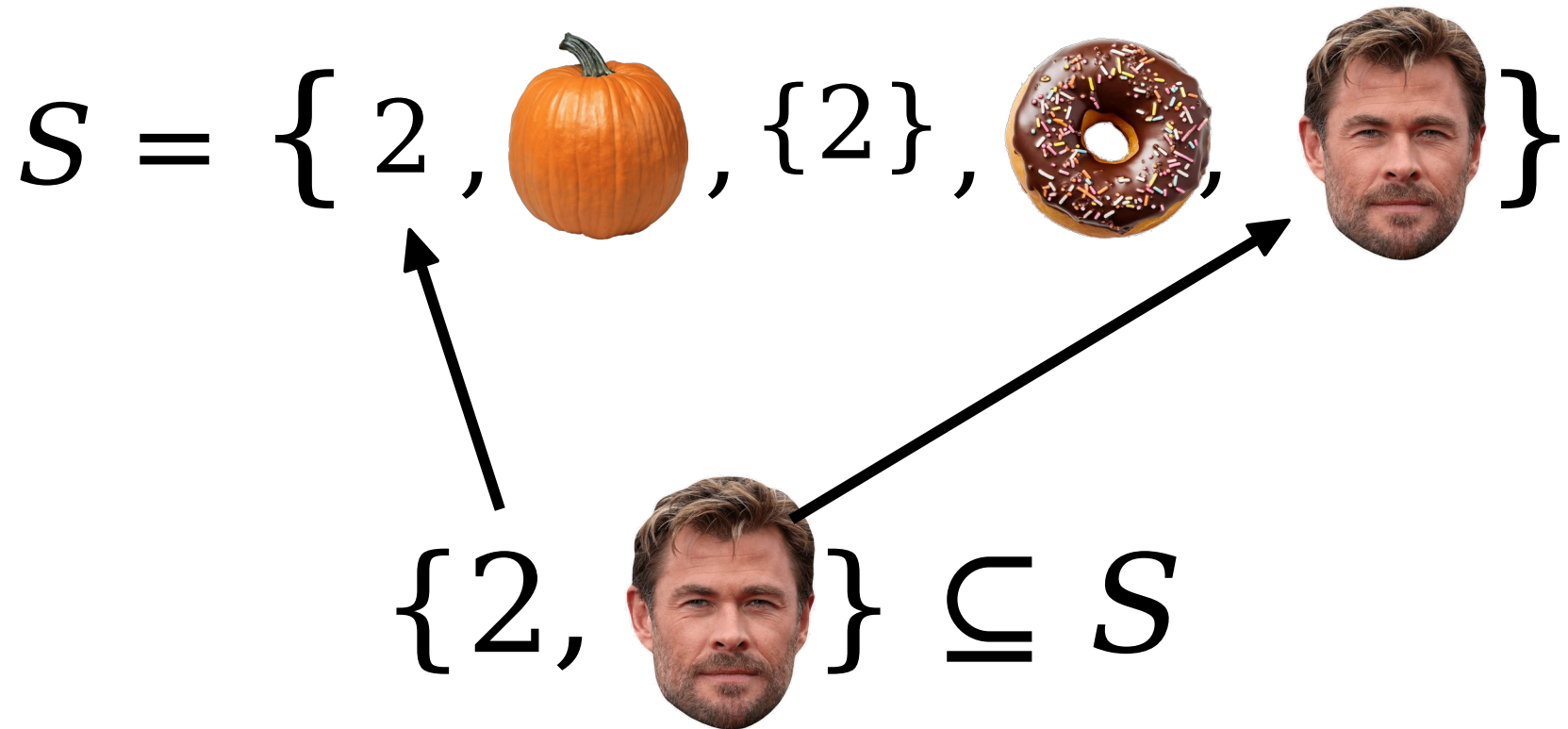
$$S = \{ 2, \text{🍂}, \{2\}, \text{🍩}, \text{👤} \}$$

$$\{2, \text{👤}\} \overset{?}{\subseteq} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$\{2, \text{Brad Pitt}\} \subseteq S$



The diagram illustrates the concept of a subset. The set S is defined as $S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$. Below it, a subset is shown as $\{2, \text{Brad Pitt}\} \subseteq S$. Two arrows originate from the subset: one points to the element '2' in the set S , and the other points to the element 'Brad Pitt' (represented by a head image) in the set S .

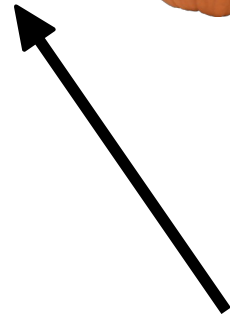
Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\{2\} \overset{?}{\subseteq} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$



$$\{2\} \subseteq S$$

Elements and Subsets

$$S = \{ 2, \text{🍂}, \{2\}, \text{🍩}, \text{👤} \}$$

$$\{\text{🍩}\} \overset{?}{\subseteq} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\{ \text{donut} \} \subseteq S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$2 \overset{?}{\subseteq} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$2 \not\in S$$

(Since 2
isn't a set.)

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

General intuition:
 $A \subseteq B$ if you can
form A by ***circling
elements of B .***

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\{2, \text{Brad Pitt}\} \subseteq S$$

General intuition:
 $A \subseteq B$ if you can
form A by ***circling
elements of B .***

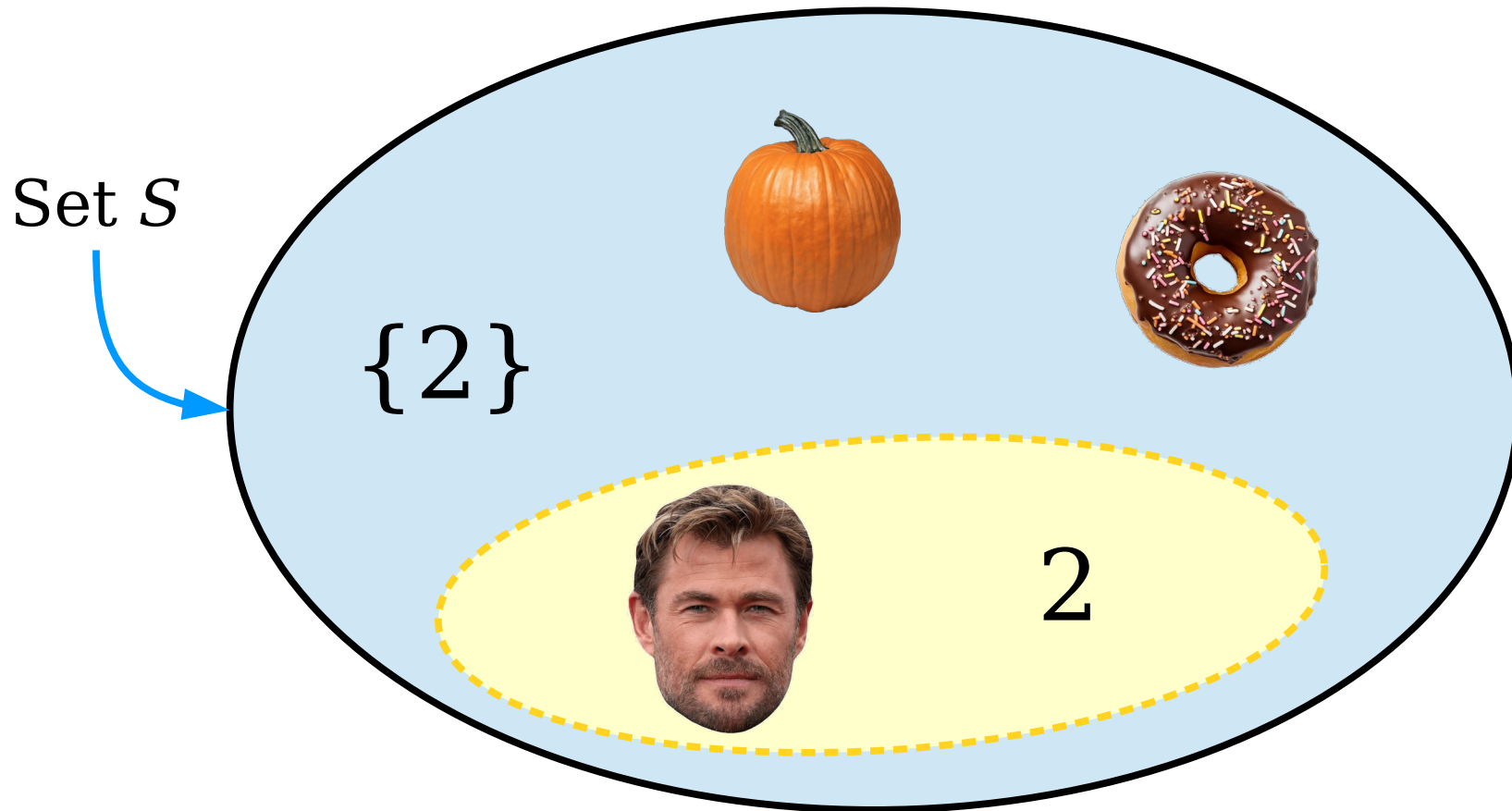
Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\{2, \text{Brad Pitt}\} \subseteq S$$

General intuition:
 $A \subseteq B$ if you can
form A by ***circling
elements of B .***

Elements and Subsets



$$\{2, \text{man}\} \subseteq S$$

General intuition:
 $A \subseteq B$ if you can
form A by ***circling***
elements of B .

Elements and Subsets

- We say that $S \in T$ when, among the elements of T , one of them is *exactly* the object S .
- We say that $S \subseteq T$ when S is a set and every element of S is also an element of T . (S has to be a set for the statement $S \subseteq T$ to be true.)
- Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.
- We have a resource on the course website, the Guide to Elements and Subsets, that explores this in more depth.

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)

2. Subsets

3. The Empty Set

4. Set Builder Notation and Some Important Sets

5. Set Operations

6. Power Sets

7. Cardinality

8. Cantor's Theorem

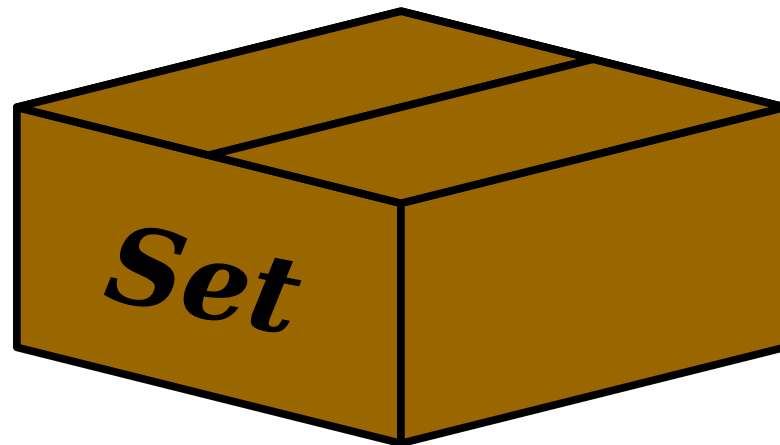
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
- 3. The Empty Set**
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

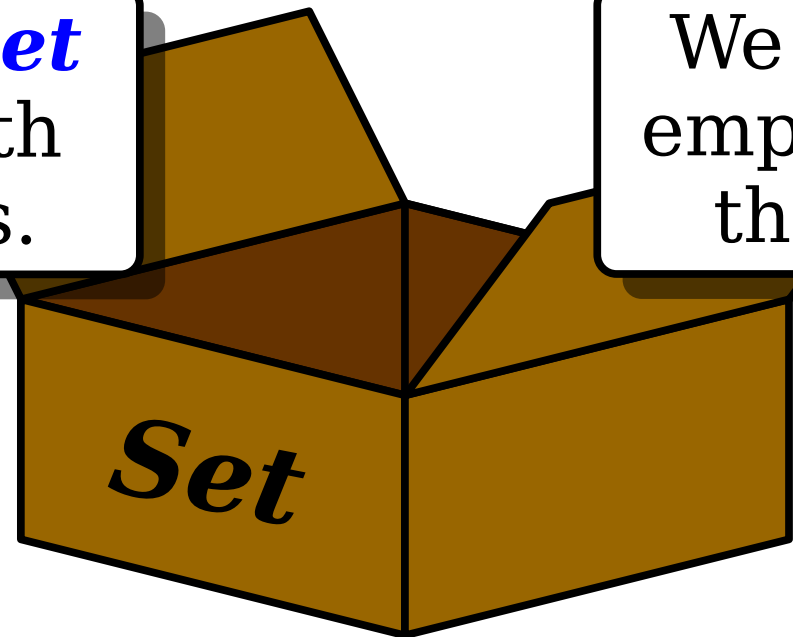


Sets can contain any number of elements.

$$\{\} = \emptyset$$

The *empty set* is the set with no elements.

We denote the empty set using this symbol.

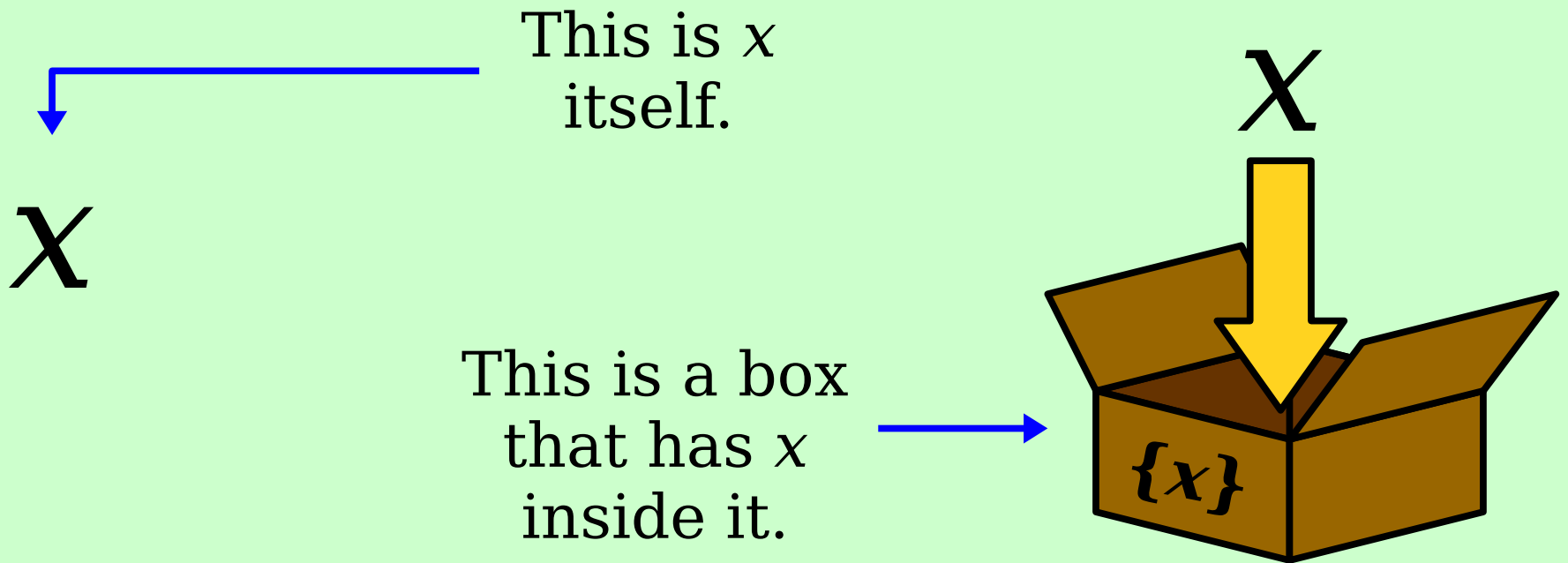


Sets can contain any number of elements.

$$\emptyset \stackrel{?}{=} \{\emptyset\}$$

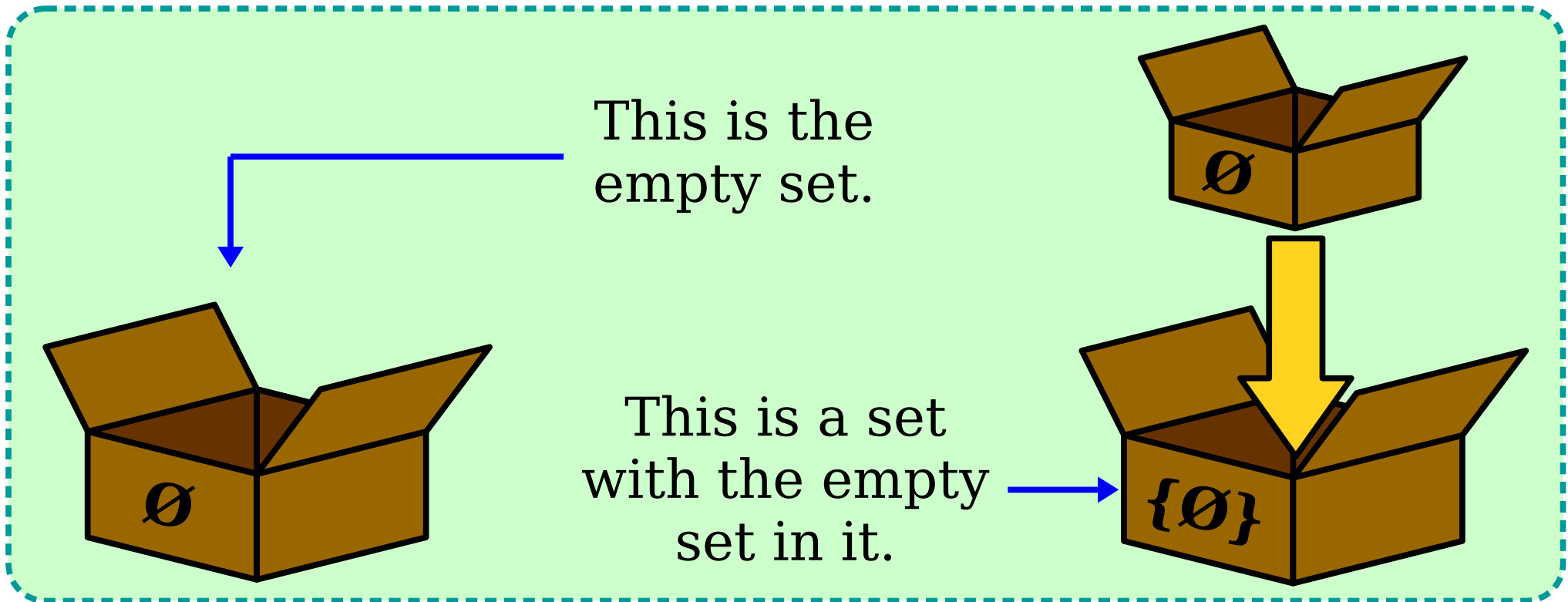
Question: Are these objects equal?

$$x \neq \{x\}$$



No object x is equal to the set containing x .

$$\emptyset \neq \{\emptyset\}$$



No object x is equal to the set containing x .

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\emptyset \overset{?}{\in} S$$

Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\emptyset \notin S$$

General intuition:
 $x \in S$ means you
can ***point at x***
inside of S.

Elements and Subsets

$$S = \{ 2, \text{🍂}, \{2\}, \text{🍩}, \text{👤} \}$$

$$\emptyset \overset{?}{\subseteq} S$$

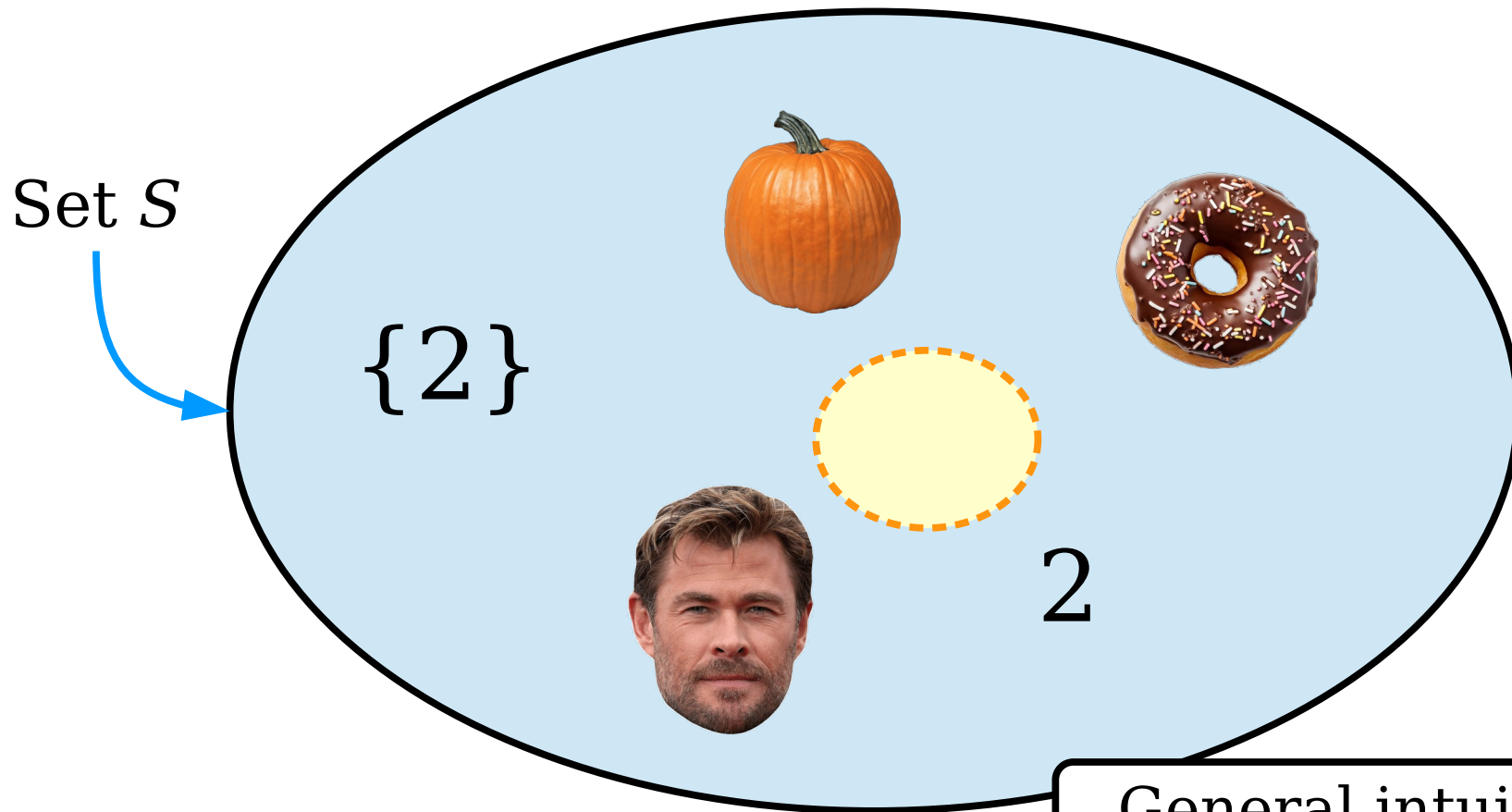
Elements and Subsets

$$S = \{ 2, \text{pumpkin}, \{2\}, \text{donut}, \text{Brad Pitt} \}$$

$$\emptyset \subseteq S$$

General intuition:
 $A \subseteq B$ if you can
form A by ***circling
elements of B .***

Elements and Subsets



$$\emptyset \subseteq S$$

General intuition:
 $A \subseteq B$ if you can
form A by ***circling
elements of B .***

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
- 3. The Empty Set**
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
- 4. Set Builder Notation and Some Important Sets**
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Infinite Sets

- Some sets contain *infinitely many* elements!
- The set $\mathbb{N} = \{ 0, 1, 2, 3, \dots \}$ is the set of all the ***natural numbers***.
 - Some mathematicians don't include zero; in this class, assume that 0 is a natural number.
- The set $\mathbb{Z} = \{ \dots, -2, -1, 0, 1, 2, \dots \}$ is the set of all the ***integers***.
 - $\mathbb{Z}$ is from German “Zahlen.”
- The set $\mathbb{R}$ is the set of all ***real numbers***.
 - $e \in \mathbb{R}$, and $4 \in \mathbb{R}$, and $-137 \in \mathbb{R}$.
- A few subset relationships:
 - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
 - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

Describing Complex Sets

- Here are some English descriptions of infinite sets:
 - “The set of all even natural numbers.”
 - “The set of all real numbers less than 137.”
 - “The set of all Python programs.”
- To describe complex sets like these mathematically, we'll use ***set-builder notation***.

Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

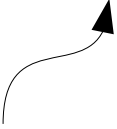
Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

Even Natural Numbers

$$\{ \textcolor{brown}{n} \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

The set of all n



Even Natural Numbers

$$\{ \textcolor{brown}{n} \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

The set of all n

where

Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

The set of all n

where

n is a natural
number

Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

The set of all n

where

n is a natural
number

and n is even

Even Natural Numbers

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

The set of all n

where

n is a natural
number

and n is even

$$\{ 0, 2, 4, 6, 8, 10, 12, 14, 16, \dots \}$$

Set Builder Notation

- A set may be specified in ***set-builder notation***:

$$\{ x \mid \text{some property } x \text{ satisfies} \}$$

$$\{ x \in S \mid \text{some property } x \text{ satisfies} \}$$

- For example:

$$\{ n \mid n \in \mathbb{N} \text{ and } n \text{ is even} \}$$

$$\{ C \mid C \text{ is a set of US coins} \}$$

$$\{ r \in \mathbb{R} \mid r < 3 \}$$

$$\{ n \in \mathbb{N} \mid n < 3 \} \quad (\text{the set } \{0, 1, 2\})$$

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
- 4. Set Builder Notation and Some Important Sets**
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

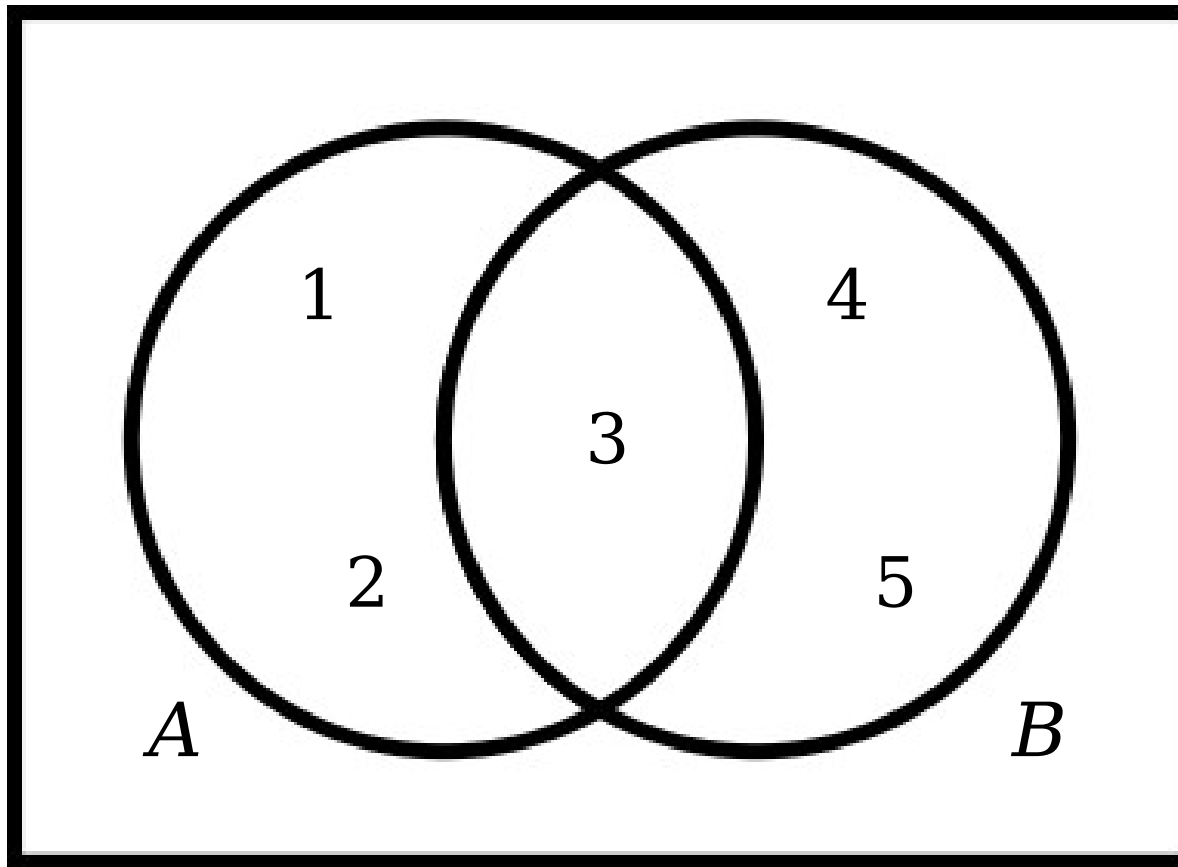
Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
- 5. Set Operations**
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

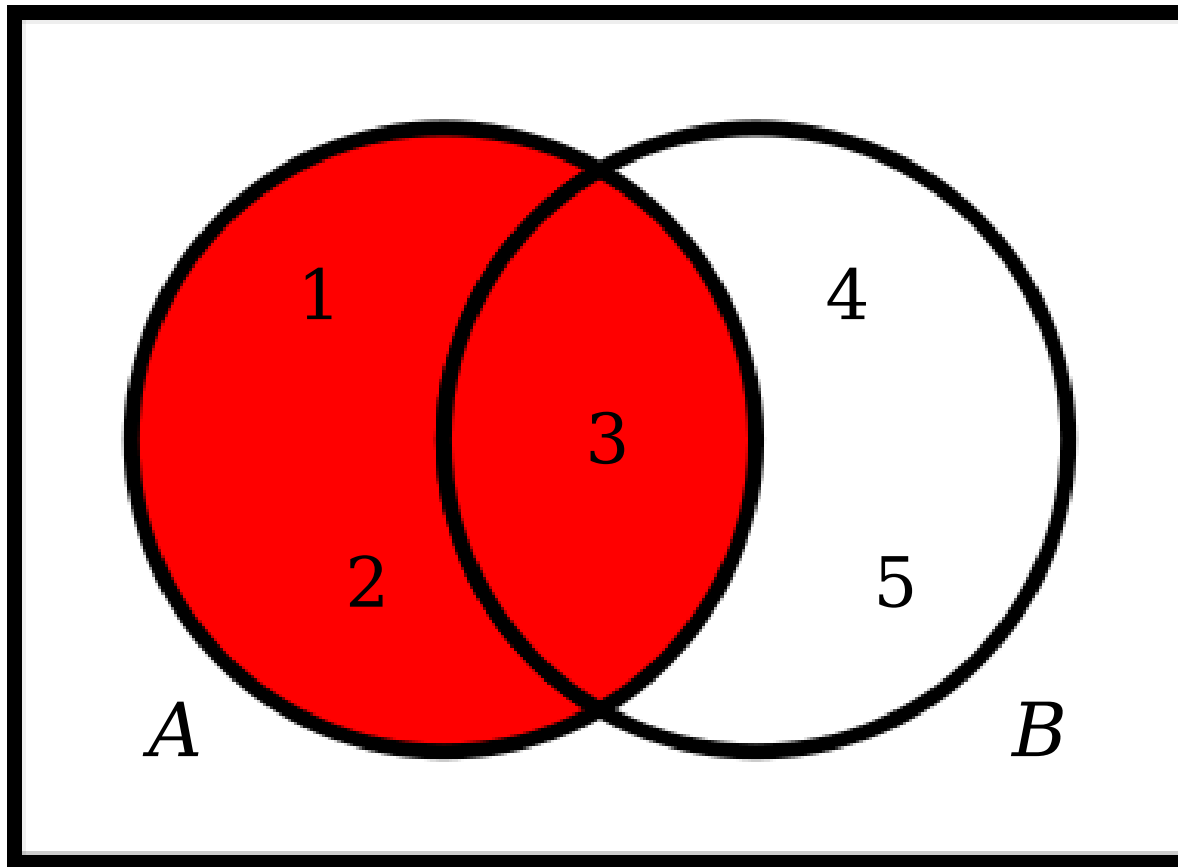
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

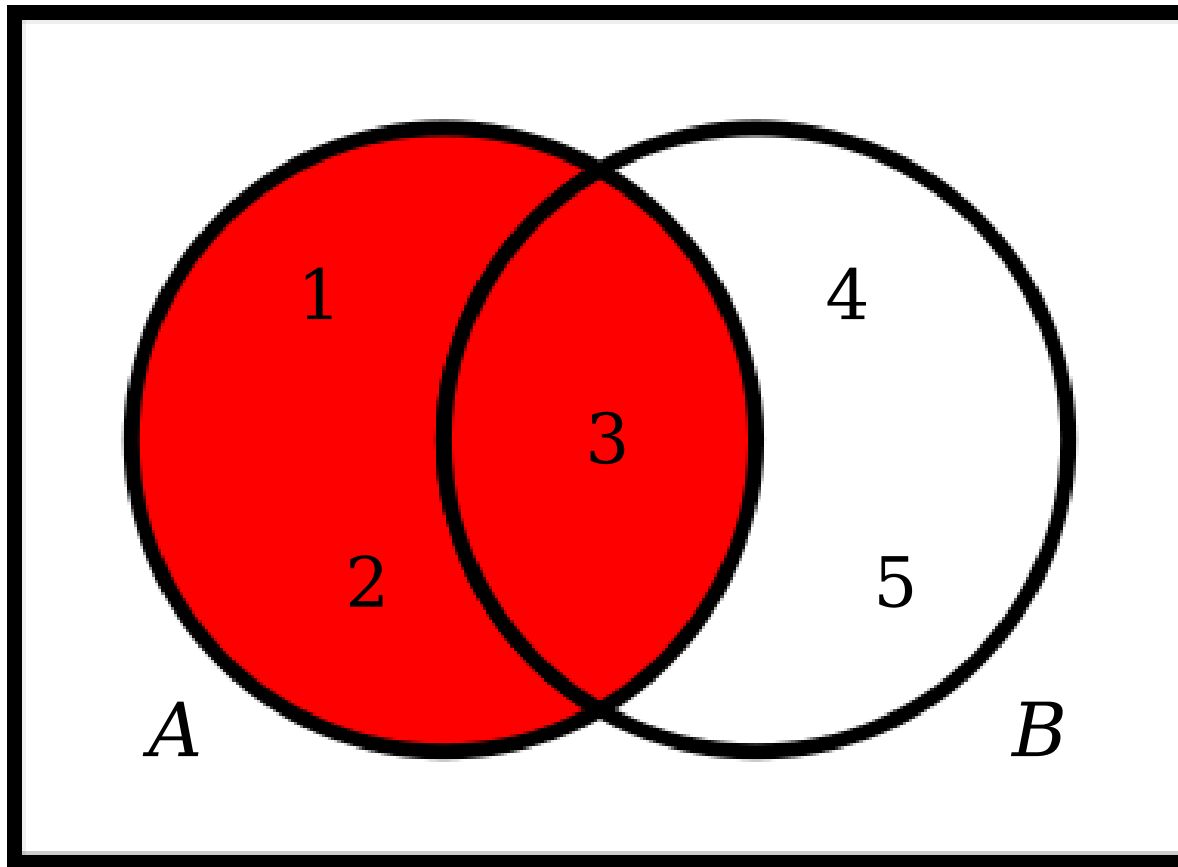
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams

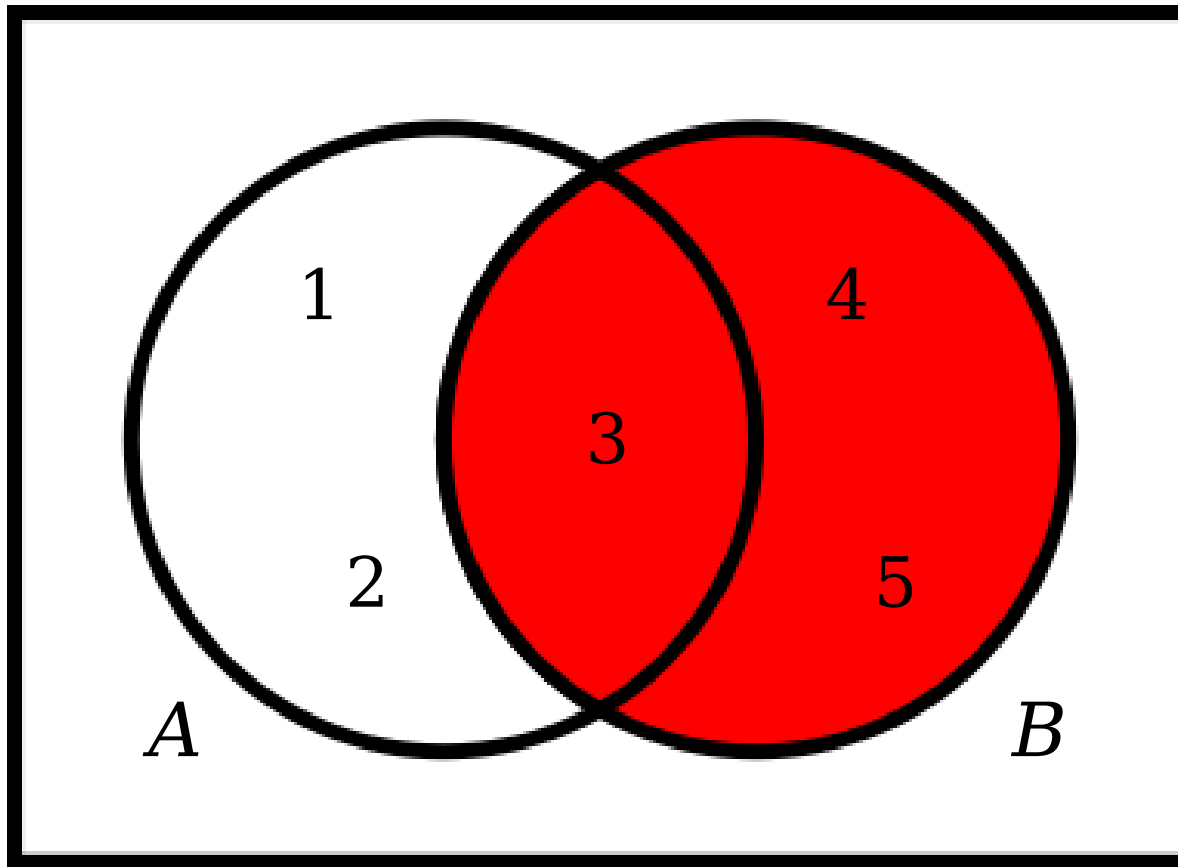


A

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

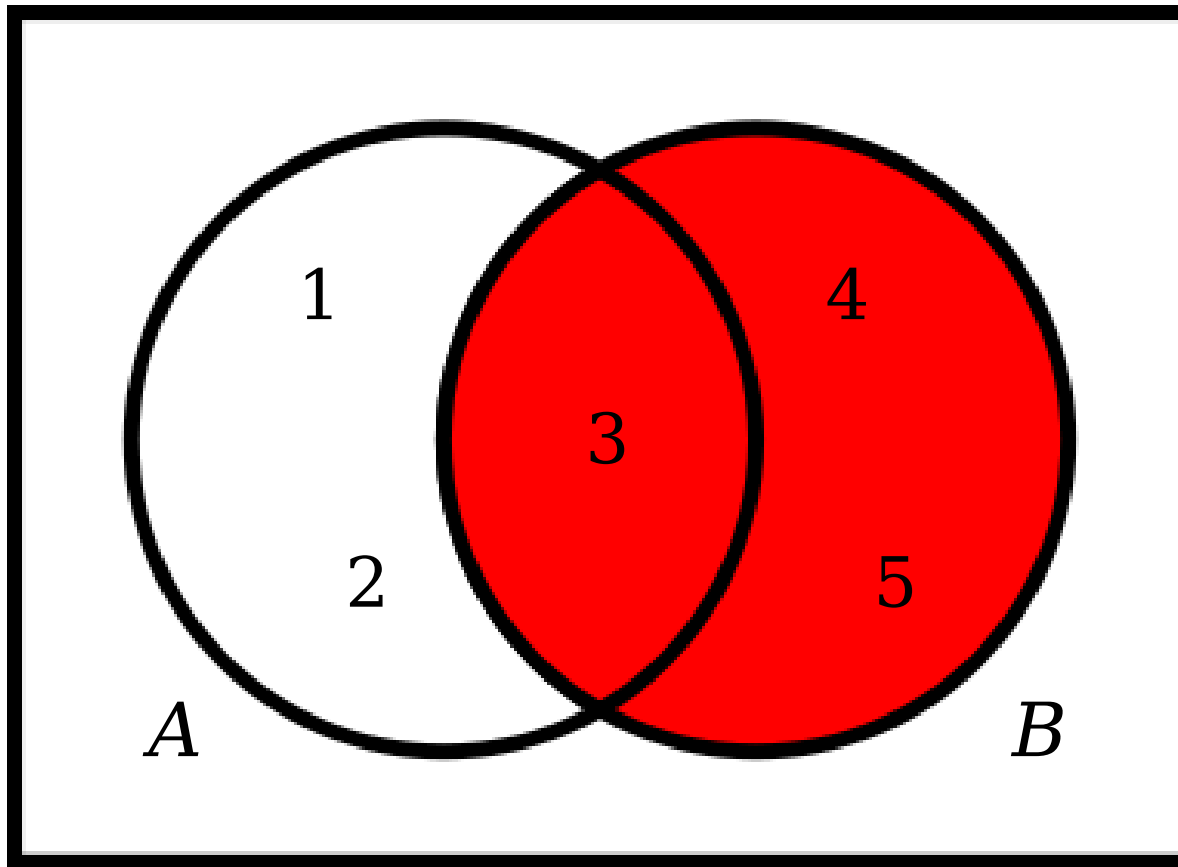
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams

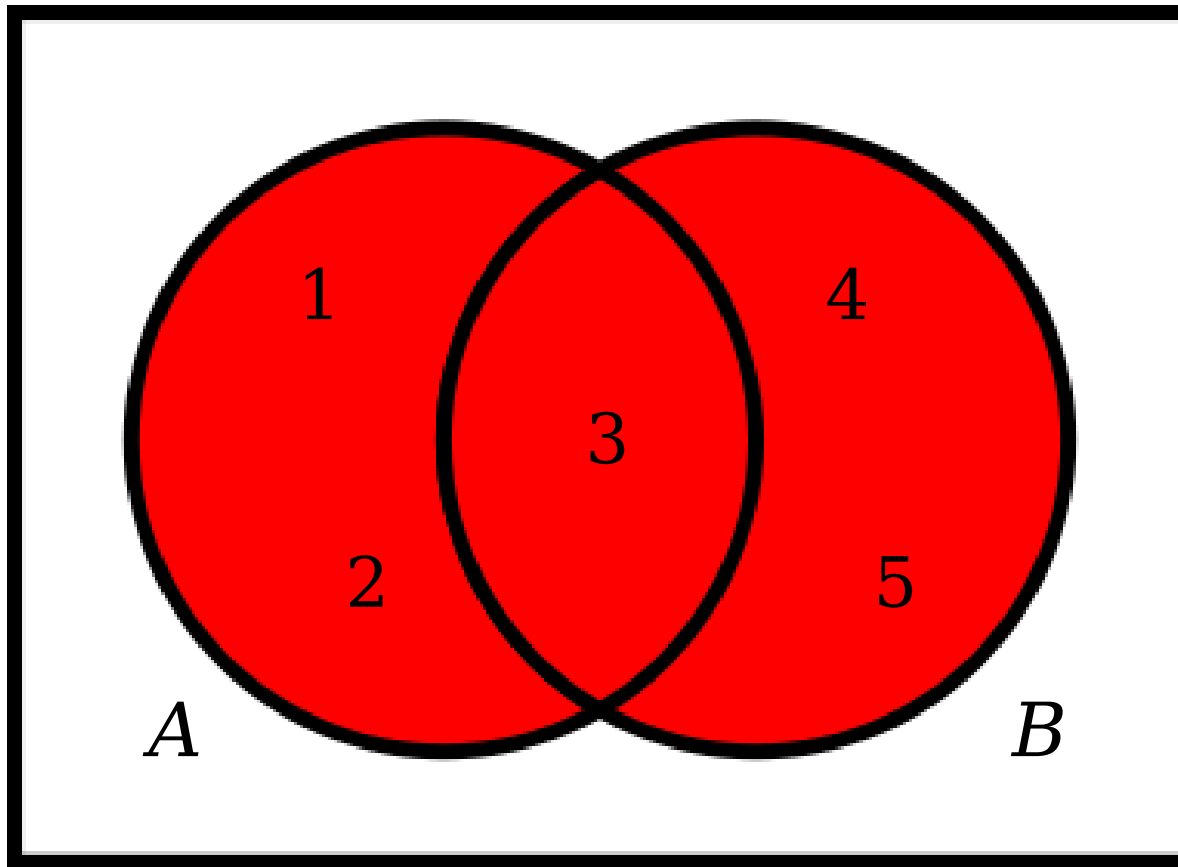


B

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

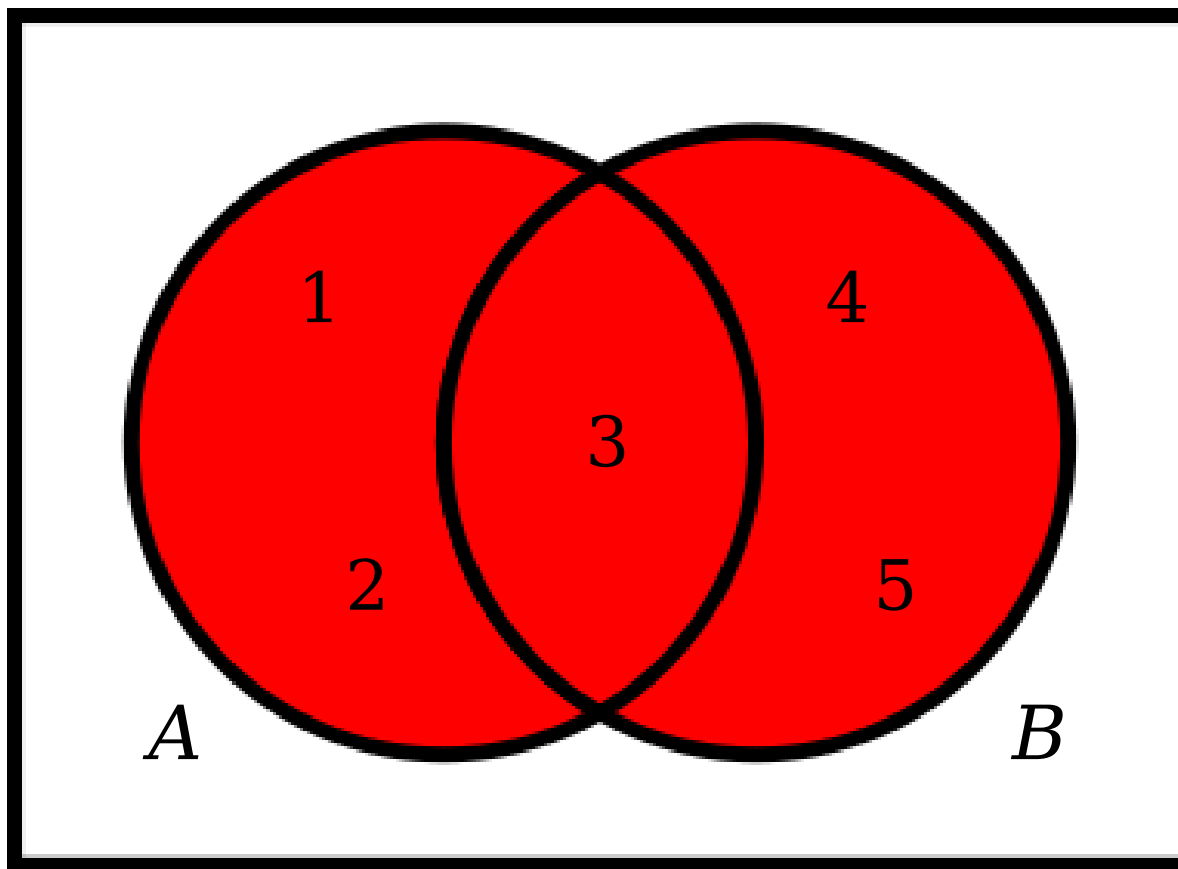
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

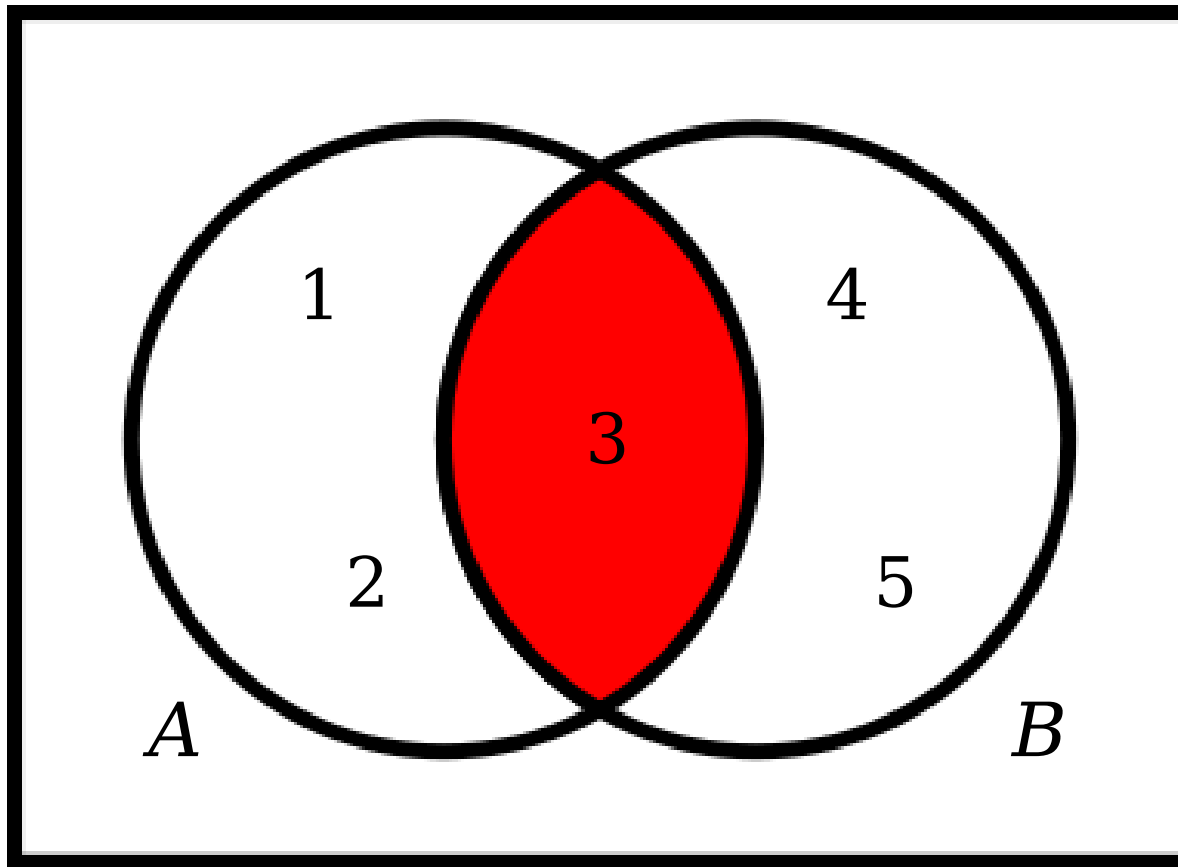
Venn Diagrams



Union
 $A \cup B$
 $\{ 1, 2, 3, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$
$$B = \{ 3, 4, 5 \}$$

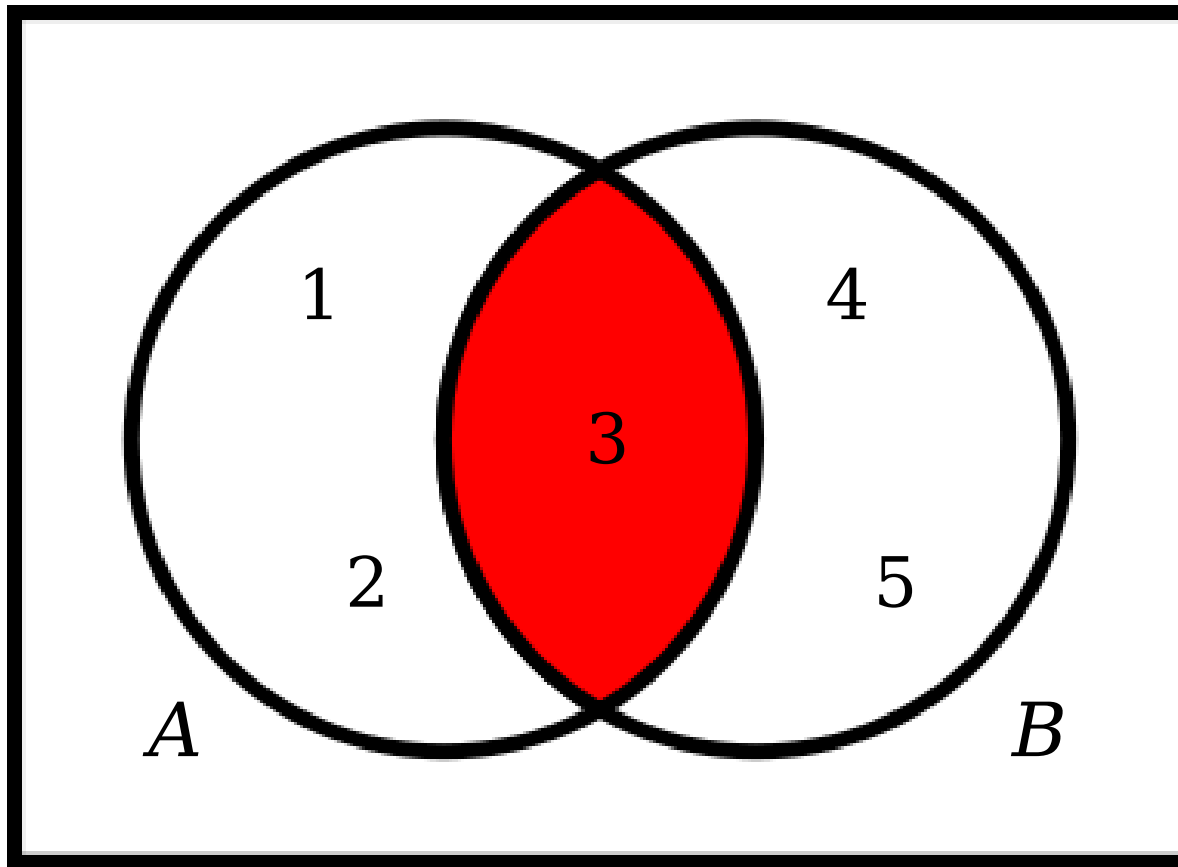
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



Intersection

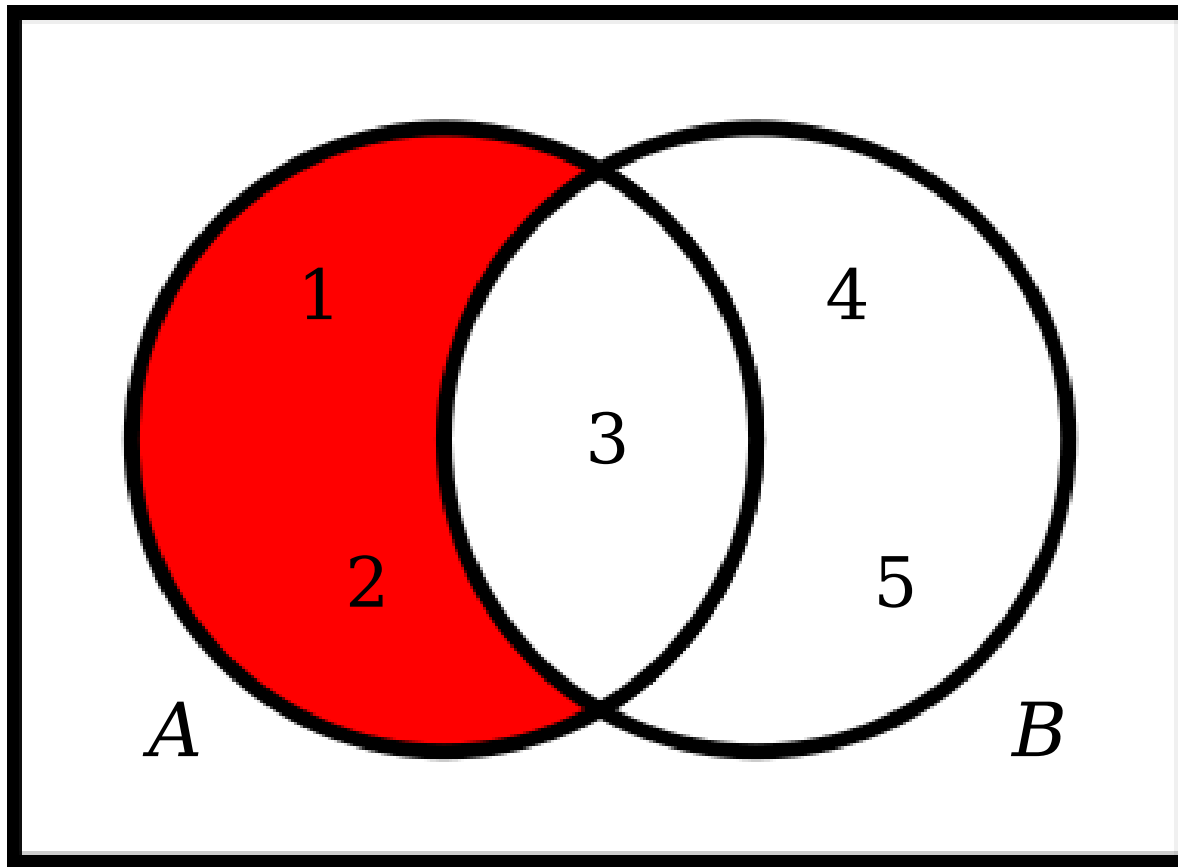
$$A \cap B$$

$$\{ 3 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

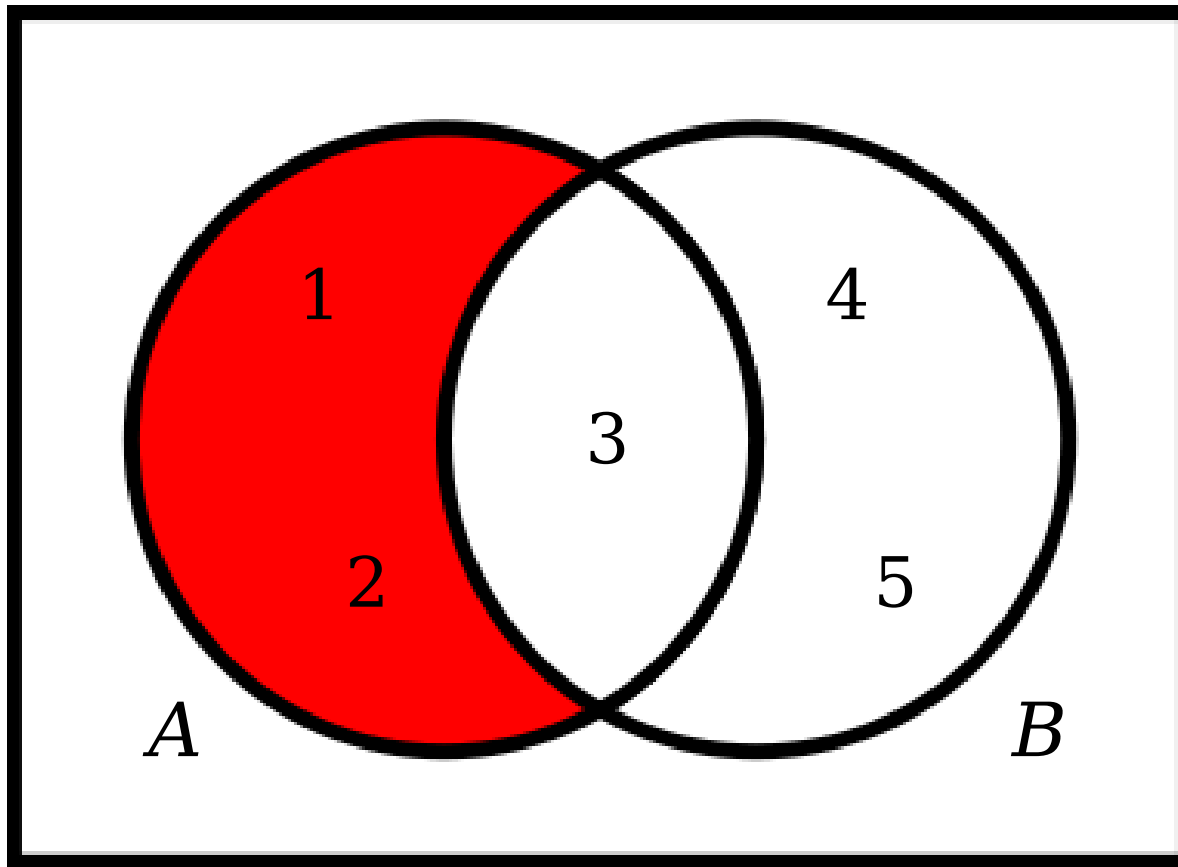
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



Difference

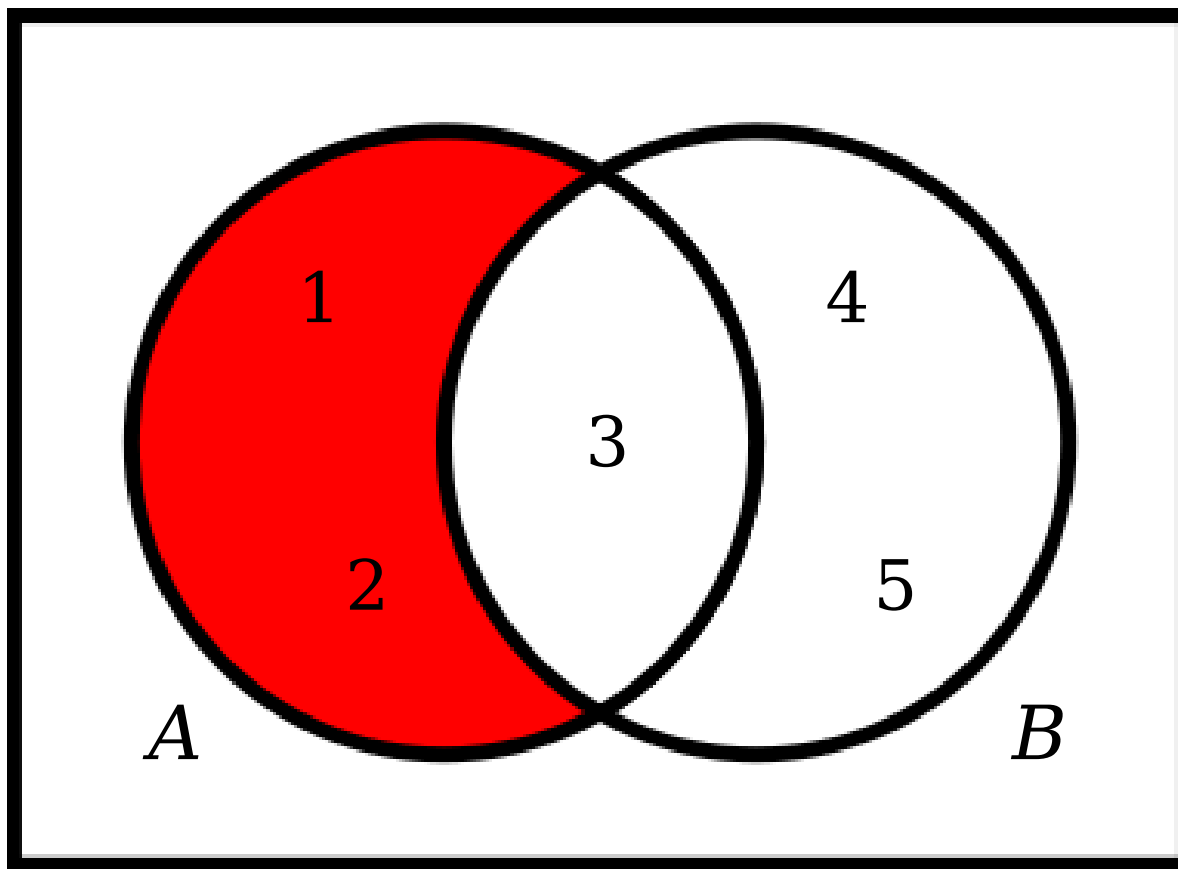
$$A - B$$

$$\{ 1, 2 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



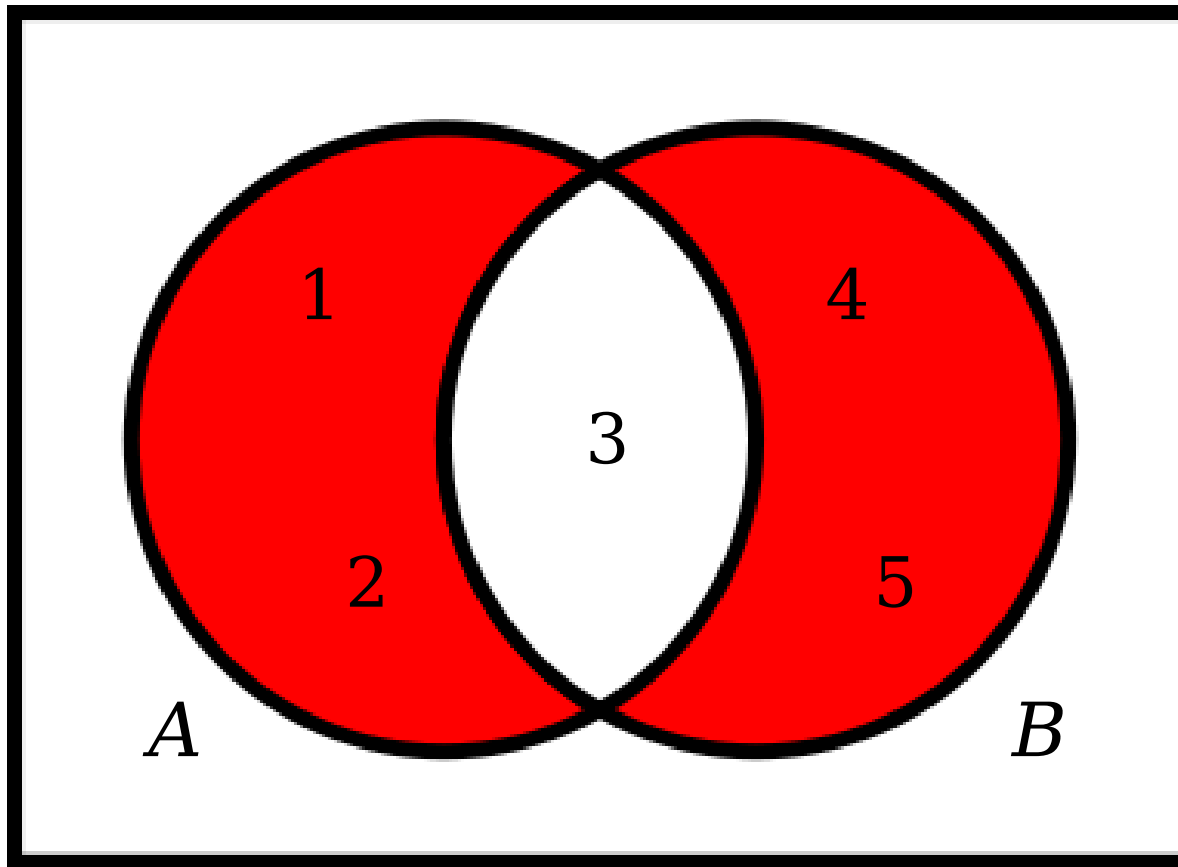
Difference

$$A \setminus B$$
$$\{ 1, 2 \}$$

$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

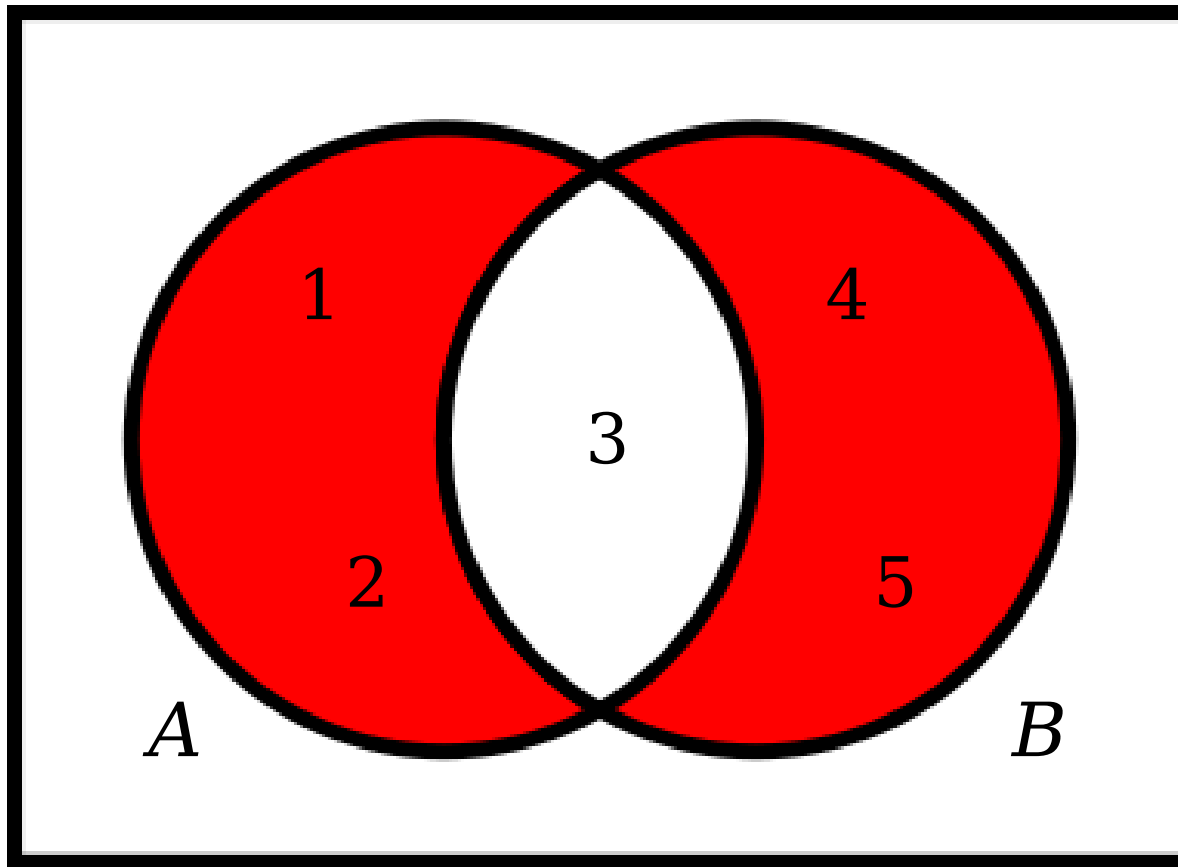
Venn Diagrams



$$A = \{ 1, 2, 3 \}$$

$$B = \{ 3, 4, 5 \}$$

Venn Diagrams



Symmetric
Difference
 $A \Delta B$
 $\{ 1, 2, 4, 5 \}$

$$A = \{ 1, 2, 3 \}$$
$$B = \{ 3, 4, 5 \}$$

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
- 5. Set Operations**
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
- 6. Power Sets**
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Power Set

The ***power set*** of S is the set of all subsets of S .
We write the power set of S as $\wp(S)$.

Power Set

The **power set** of S is the set of all subsets of S .
We write the power set of S as $\wp(S)$.

Power Set

$$S = \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\}$$

The **power set** of S is the set of all subsets of S .
We write the power set of S as $\wp(S)$.

Power Set

$$S = \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\}$$

$$\wp(S) =$$

The **power set** of S is the set of all subsets of S .
We write the power set of S as $\wp(S)$.

Power Set

$$S = \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\}$$

$$\wp(S) = \emptyset \quad \left\{ \text{pumpkin} \right\} \quad \left\{ \text{Chris Hemsworth} \right\} \quad \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\}$$

The **power set** of S is the set of all subsets of S .
We write the power set of S as $\wp(S)$.

Power Set

$$S = \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\}$$

$$\wp(S) = \left\{ \emptyset, \left\{ \text{pumpkin} \right\}, \left\{ \text{Chris Hemsworth} \right\}, \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\} \right\}$$

The **power set** of S is the set of all subsets of S .
We write the power set of S as $\wp(S)$.

Power Set

$$S = \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\}$$

$$\wp(S) = \left\{ \emptyset, \left\{ \text{pumpkin} \right\}, \left\{ \text{Chris Hemsworth} \right\}, \left\{ \text{pumpkin}, \text{Chris Hemsworth} \right\} \right\}$$

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$.

(Do you see why?)

The **power set** of S is the set of all subsets of S .

We write the power set of S as $\wp(S)$.

Power Set

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$!

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$.

(Do you see why?)

The **power set** of S is the set of all subsets of S .

We write the power set of S as $\wp(S)$.

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
- 6. Power Sets**
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
- 7. Cardinality**
8. Cantor's Theorem
9. Shocking Implications

Cardinality

- The **cardinality** of a set is the number of elements it contains.
- If S is a set, we denote its cardinality as $|S|$.
- Examples:
 - $|\{\textit{whimsy}, \textit{mirth}\}| = 2$
 - $|\{\{a, b\}, \{c, d, e, f, g\}, \{h\}\}| = 3$
 - $|\{1, 2, 3, 3, 3, 3, 3\}| = 3$
 - $|\{n \in \mathbb{N} \mid n < 4\}| = |\{0, 1, 2, 3\}| = 4$
 - $|\emptyset| = 0$
 - $|\{\emptyset\}| = 1$

The Cardinality of $\mathbb{N}$

- What is $|\mathbb{N}|$?
 - There are infinitely many natural numbers.
 - $|\mathbb{N}|$ can't be a natural number, since it's infinitely large.
- We need to introduce a new term.
- Let's define $\aleph_0 = |\mathbb{N}|$.
 - $\aleph_0$ is pronounced “aleph-zero,” “aleph-nought,” or “aleph-null.”
- **Question:** Why don't we say $|\mathbb{N}| = \infty$?

Cardinality of Infinite Sets

Astonishing Fact:

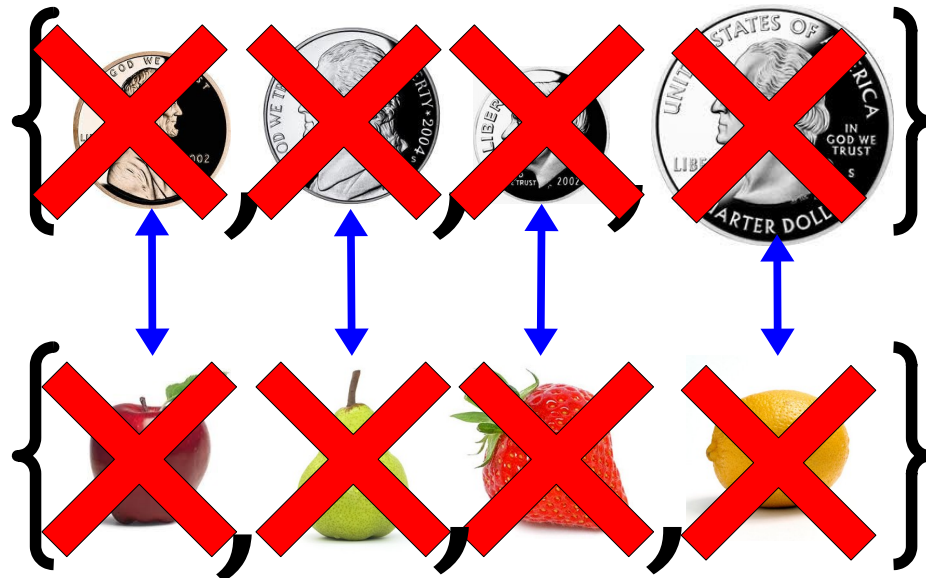
Not all infinite sets have the same cardinality.
Some infinite sets are bigger than others!

More Astonishing Fact:

This has practical consequences!

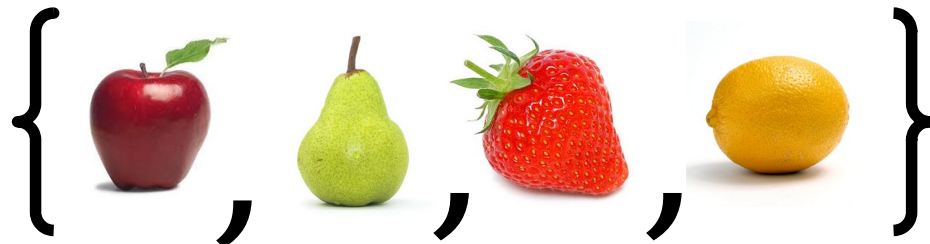
Comparing Cardinalities

- **Question:** What does it mean for one set to be “bigger” than another?
- If S and T are sets, we say that $|S| = |T|$ when there is a way of pairing off the elements of S and T without leaving anything uncovered.



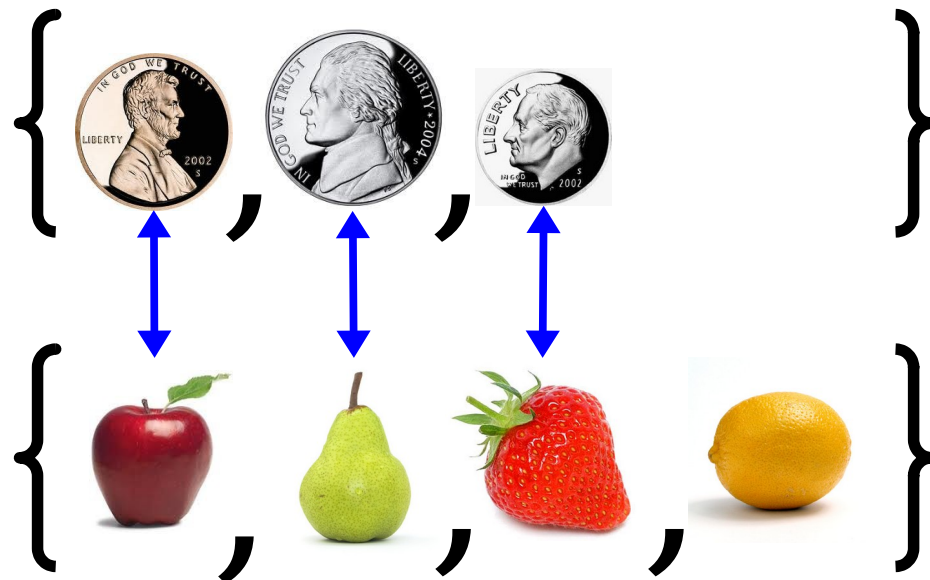
Comparing Cardinalities

- **Question:** What does it mean for one set to be “bigger” than another?
- If S and T are sets, we say that $|S| = |T|$ when there is a way of pairing off the elements of S and T without leaving anything uncovered.



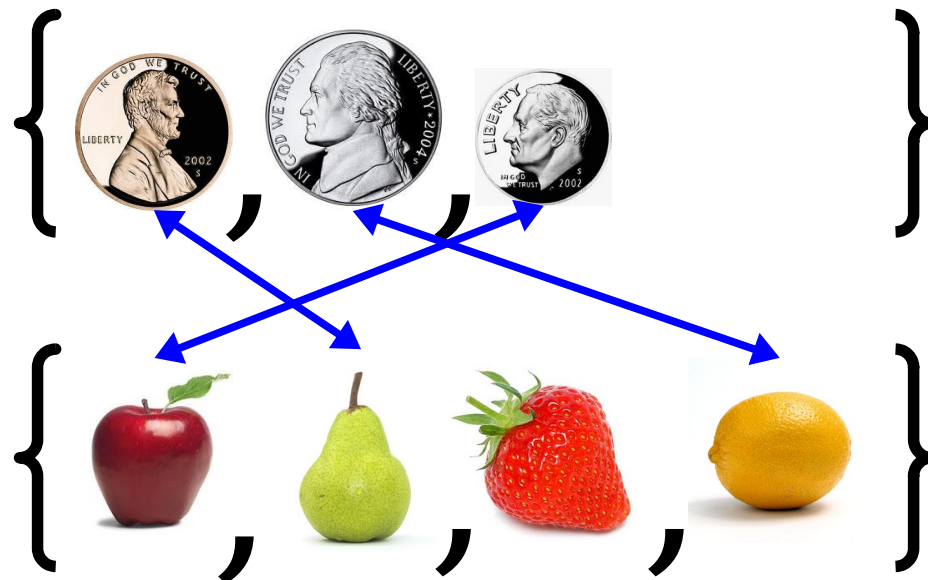
Comparing Cardinalities

- **Question:** What does it mean for one set to be “bigger” than another?
- If S and T are sets, we say that $|S| = |T|$ when there is a way of pairing off the elements of S and T without leaving anything uncovered.



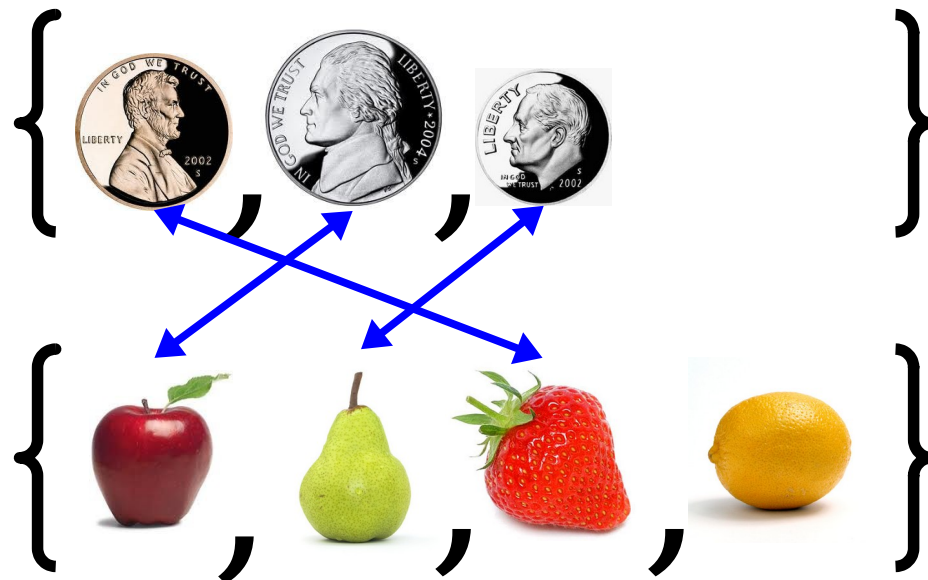
Comparing Cardinalities

- **Question:** What does it mean for one set to be “bigger” than another?
- If S and T are sets, we say that $|S| = |T|$ when there is a way of pairing off the elements of S and T without leaving anything uncovered.



Comparing Cardinalities

- **Question:** What does it mean for one set to be “bigger” than another?
- If S and T are sets, we say that $|S| = |T|$ when there is a way of pairing off the elements of S and T without leaving anything uncovered.



Infinite Cardinalities

$\mathbb{N}$ 0 1 2 3 4 5 6 7 8 ...

$\mathbb{Z}$... -3 -2 -1 0 1 2 3 4 ...

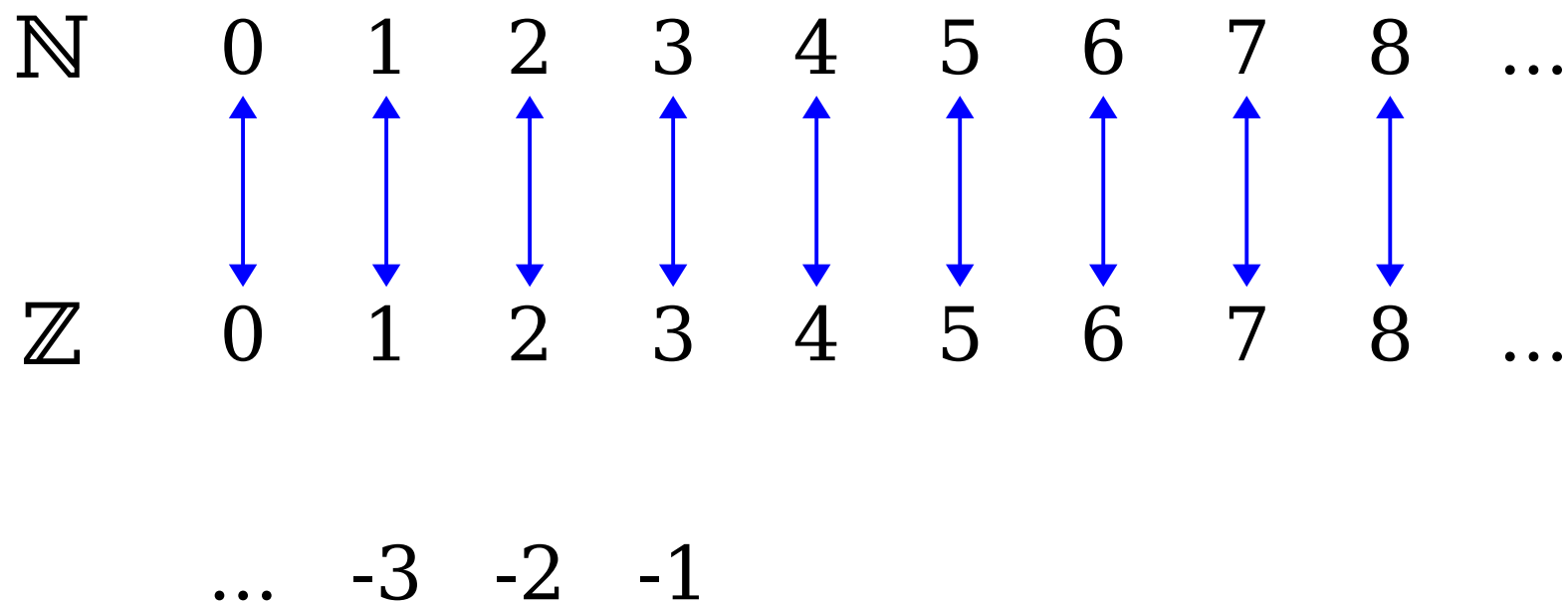
Infinite Cardinalities

$\mathbb{N}$ 0 1 2 3 4 5 6 7 8 ...

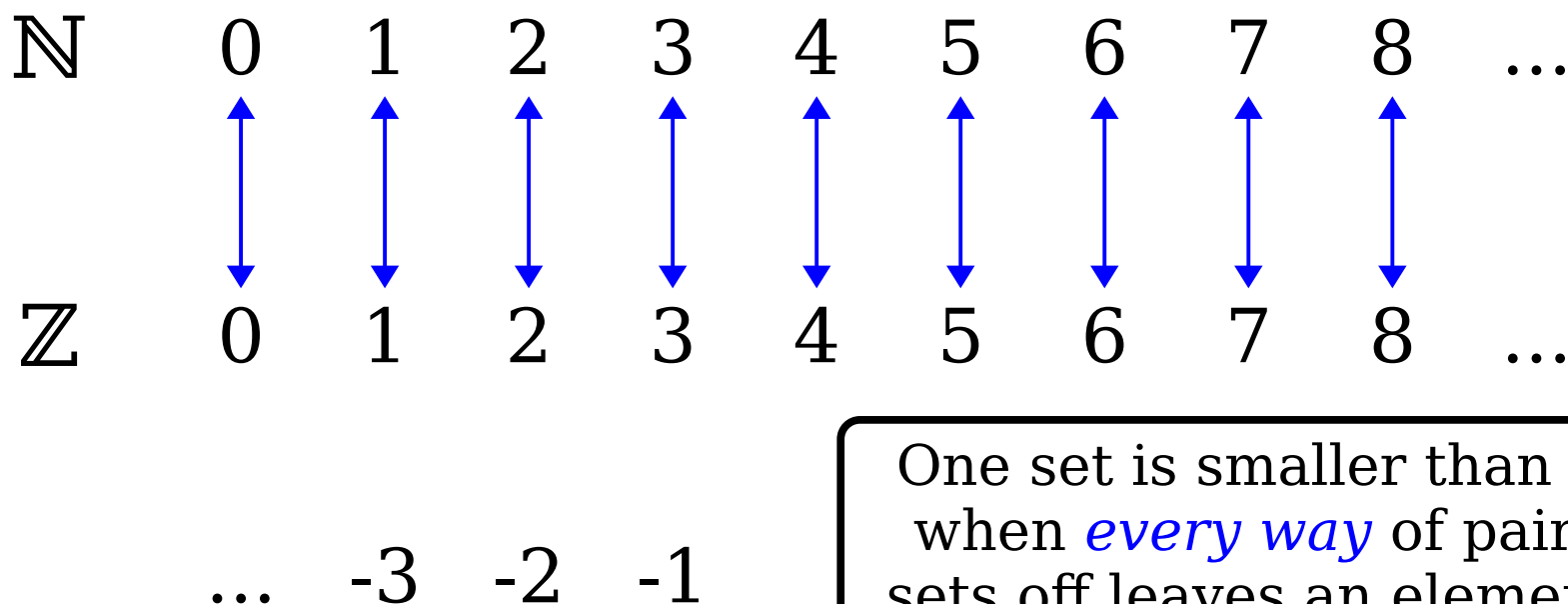
$\mathbb{Z}$ 0 1 2 3 4 ...

... -3 -2 -1

Infinite Cardinalities



Infinite Cardinalities



One set is smaller than another when *every way* of pairing the sets off leaves an element of the second set uncovered.

Two sets have the same size when *there is a way* to pair their elements off without leaving any elements uncovered

Infinite Cardinalities

$\mathbb{N}$ 0 1 2 3 4 5 6 7 8 ...

$\mathbb{Z}$... -3 -2 -1 0 1 2 3 4 ...

Infinite Cardinalities

$\mathbb{N}$ 0 1 2 3 4 5 6 7 8 ...

$\mathbb{Z}$

... -3 -2 -1 0 1 2 3 4 ...

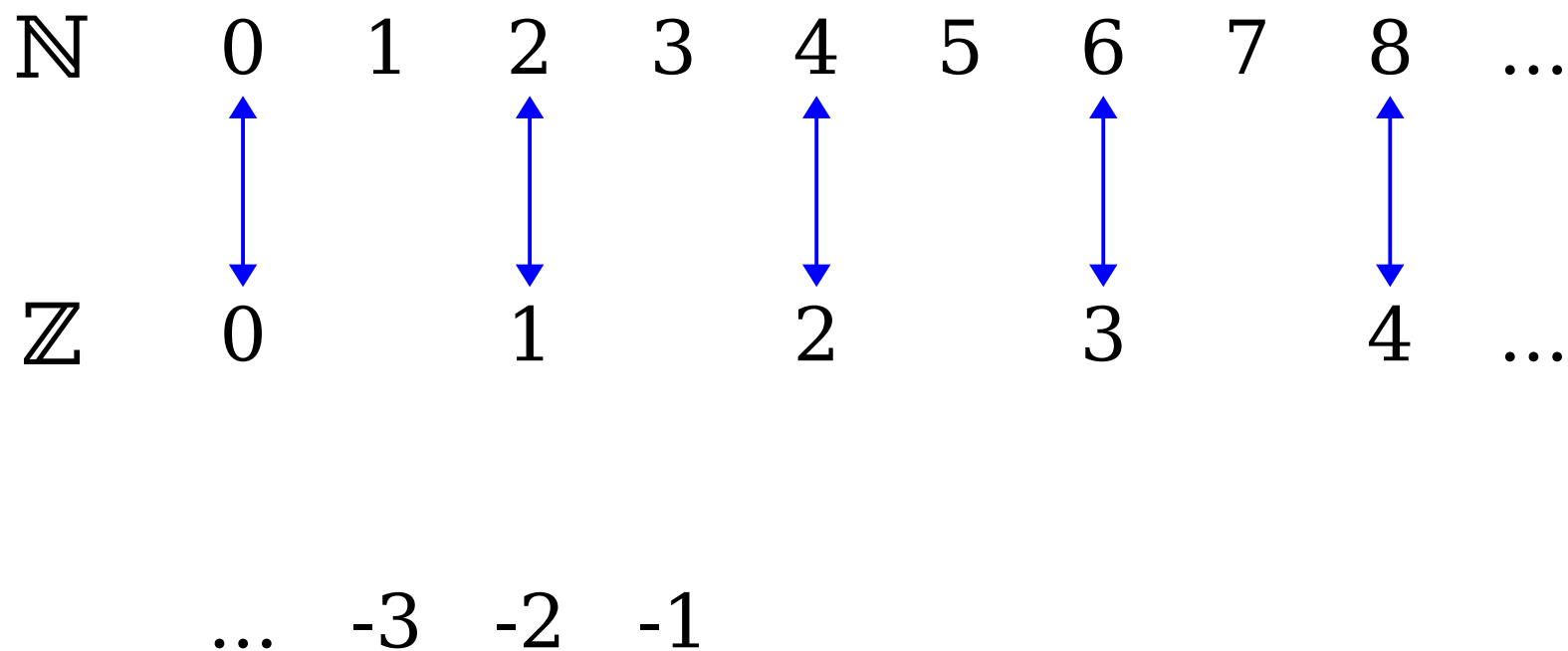
Infinite Cardinalities

$\mathbb{N}$ 0 1 2 3 4 5 6 7 8 ...

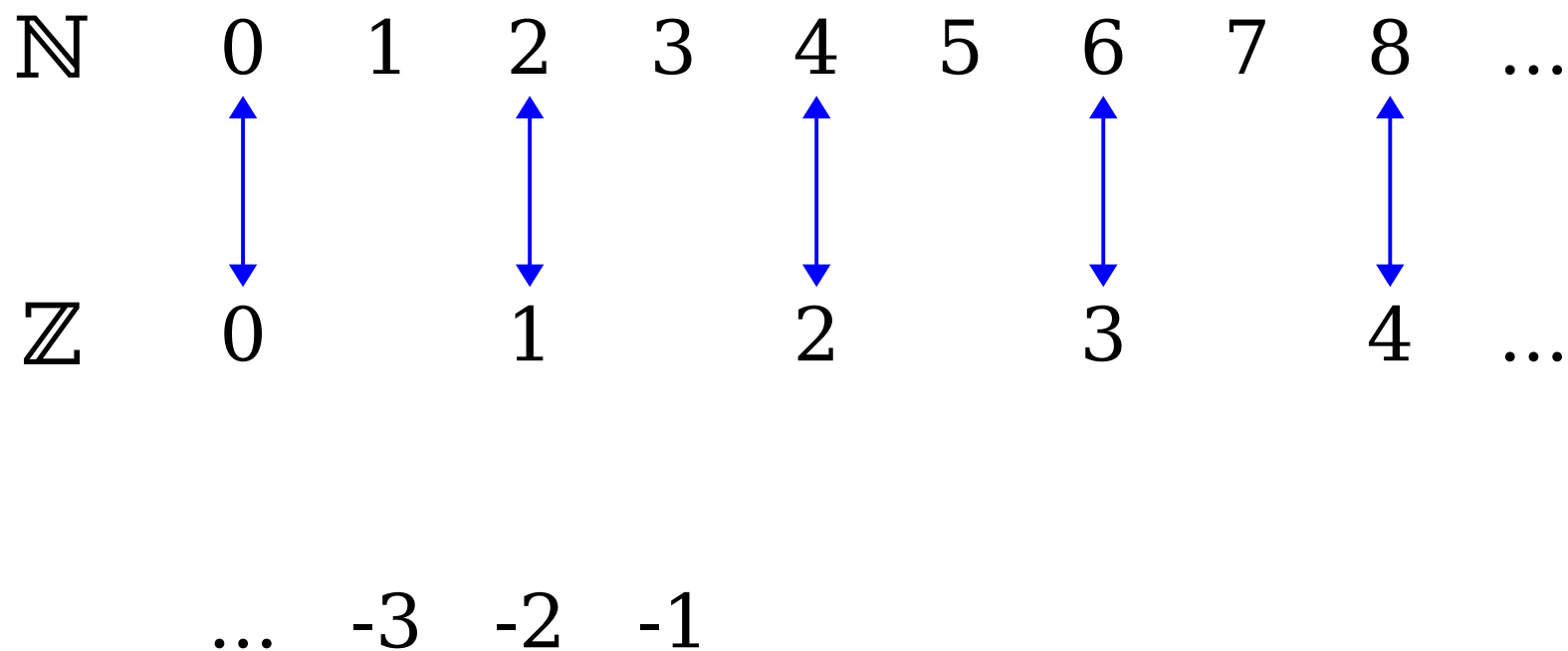
$\mathbb{Z}$ 0 1 2 3 4 ...

... -3 -2 -1

Infinite Cardinalities

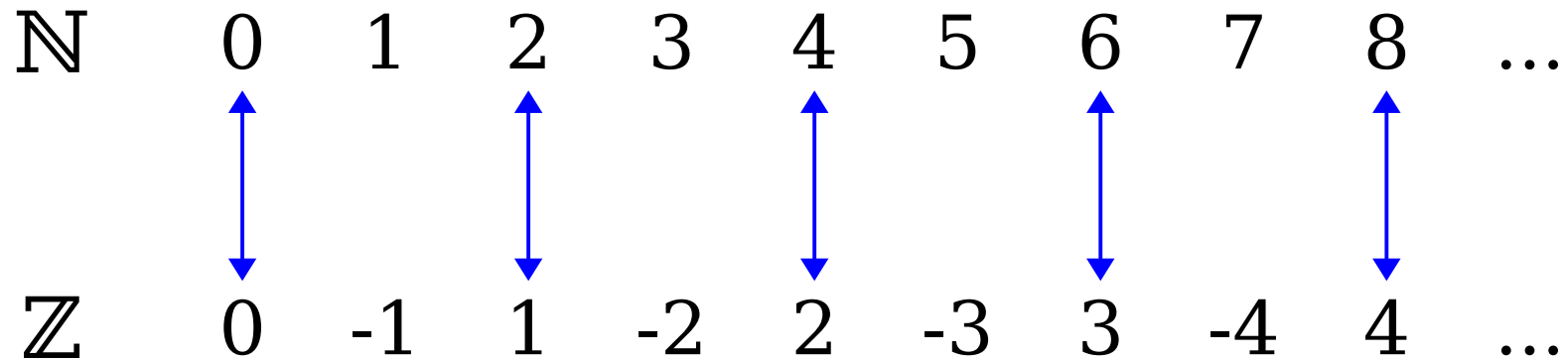


Infinite Cardinalities



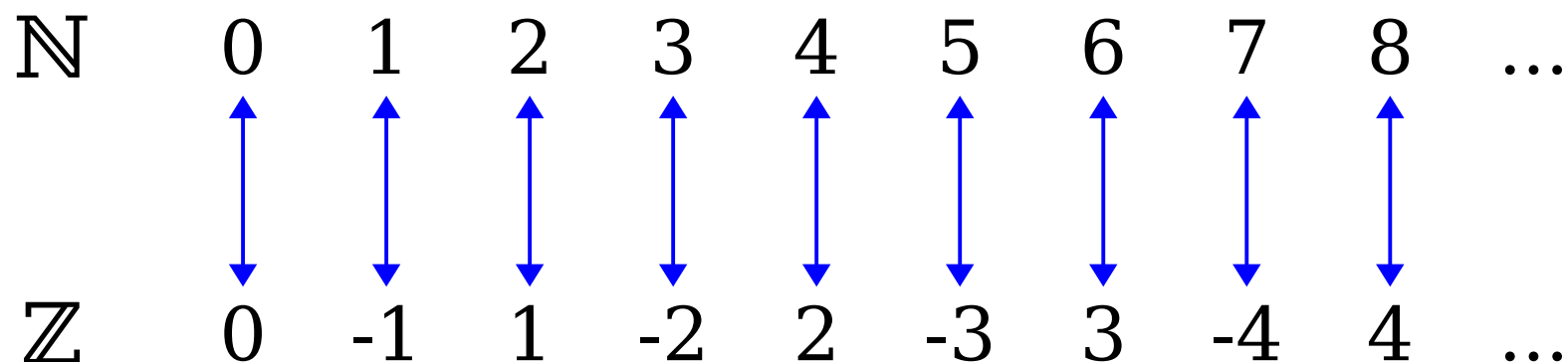
Pair nonnegative integers with even natural numbers.

Infinite Cardinalities



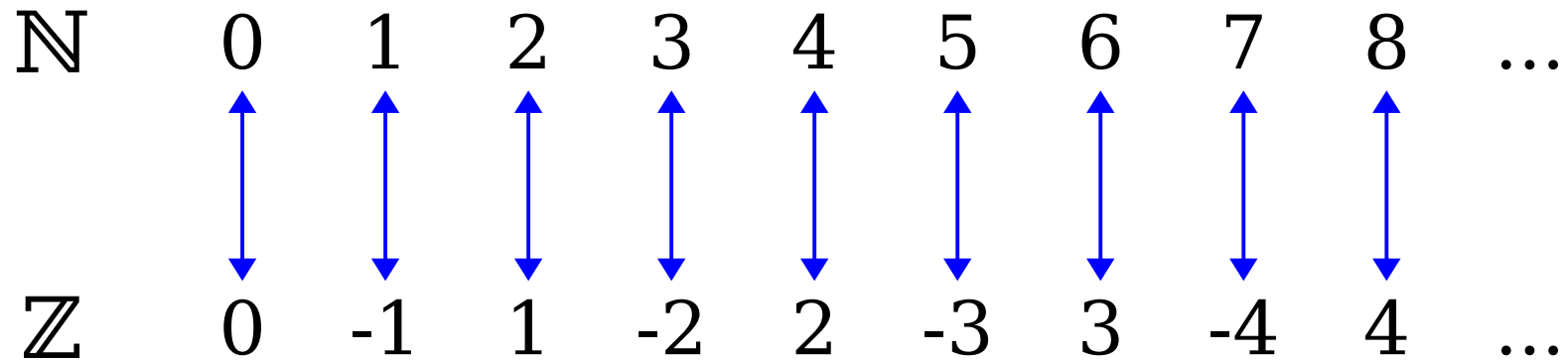
Pair nonnegative integers with even natural numbers.

Infinite Cardinalities



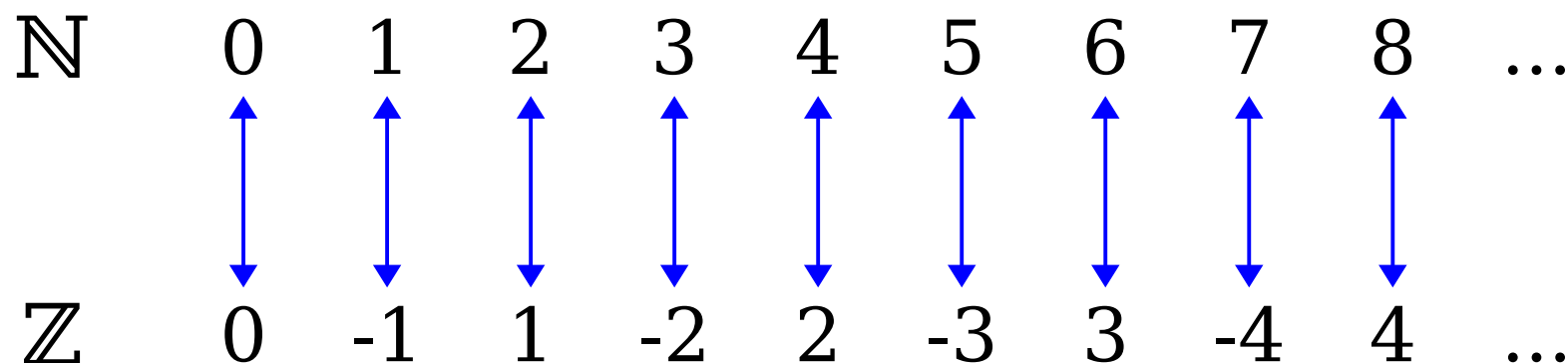
Pair nonnegative integers with even natural numbers.

Infinite Cardinalities



Pair nonnegative integers with even natural numbers.
Pair negative integers with odd natural numbers.

Infinite Cardinalities



$$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$$

Pair nonnegative integers with even natural numbers.

Pair negative integers with odd natural numbers.

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
- 7. Cardinality**
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
- 8. Cantor's Theorem**
9. Shocking Implications

Cantor's Theorem:

If S is a set, then $|S| < |\wp(S)|$.

$$S = \left\{ \text{Lincoln Penny}, \text{Jefferson Nickel} \right\}$$

$$\wp(S) = \left\{ \emptyset, \left\{ \text{Jefferson Nickel} \right\}, \left\{ \text{Lincoln Penny} \right\}, \left\{ \text{Lincoln Penny}, \text{Jefferson Nickel} \right\} \right\}$$

$$|S| < |\wp(S)|$$

$$S = \left\{ \text{Lincoln Penny}, \text{Washington Quarter}, \text{Button} \right\}$$

$$\wp(S) = \left\{ \begin{aligned} &\emptyset, \{ \text{Lincoln Penny} \}, \{ \text{Washington Quarter} \}, \{ \text{Button} \}, \\ &\{ \text{Lincoln Penny}, \text{Washington Quarter} \}, \{ \text{Lincoln Penny}, \text{Button} \}, \{ \text{Washington Quarter}, \text{Button} \}, \\ &\{ \text{Lincoln Penny}, \text{Washington Quarter}, \text{Button} \} \end{aligned} \right\}$$

$$|S| < |\wp(S)|$$

$$S = \{a, b, c, d\}$$

$$\wp(S) = \{ \emptyset, \\ \{a\}, \{b\}, \{c\}, \{d\}, \\ \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\} \\ \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \\ \{a, b, c, d\} \}$$

$$|S| < |\wp(S)|$$

Cantor's Theorem:

If S is a set, then $|S| < |\wp(S)|$.

Stated differently: no matter how you pair off the elements of a set S with the elements of $\wp(S)$, there is always some element of $\wp(S)$ left uncovered.

Cantor's Theorem:

If S is a set, then $|S| < |\wp(S)|$.

Stated differently: no matter how you pair off the elements of a set S with the **elements of $\wp(S)$** , there is always some **element of $\wp(S)$** left uncovered.

Cantor's Theorem:

If S is a set, then $|S| < |\wp(S)|$.

Stated differently: no matter how you pair off the elements of a set S with the **subsets of S** , there is always some **subset of S** left uncovered.

Cantor's Theorem:

If S is a set, then $|S| < |\wp(S)|$.

Stated differently: no matter how you pair off the elements of a set S with the subsets of S , there is always some subset of S left uncovered.

Cantor's Theorem

Consider the set

$$S = \{x_0, x_1, x_2, x_3, x_4, x_5, \dots\}.$$

Let's try pairing elements of S to elements of $\wp(S)$ and see what happens.

$$\begin{array}{lcl}
x_0 & \longleftrightarrow & \{ x_0, x_2, x_4, \dots \} \\
x_1 & \longleftrightarrow & \{ x_3, x_5, \dots \} \\
x_2 & \longleftrightarrow & \{ x_0, x_1, x_2, x_5, \dots \} \\
x_3 & \longleftrightarrow & \{ x_1, x_4, \dots \} \\
x_4 & \longleftrightarrow & \{ x_2, \dots \} \\
x_5 & \longleftrightarrow & \{ x_0, x_4, x_5, \dots \} \\
\dots & \longleftrightarrow & \{ \dots \}
\end{array}$$

x_0	x_1	x_2	x_3	x_4	x_5	$\dots$
-------	-------	-------	-------	-------	-------	---------

$$\begin{aligned}
 x_0 &\longleftrightarrow \{ x_0, x_2, x_4, \dots \} \\
 x_1 &\longleftrightarrow \{ x_3, x_5, \dots \} \\
 x_2 &\longleftrightarrow \{ x_0, x_1, x_2, x_5, \dots \} \\
 x_3 &\longleftrightarrow \{ x_1, x_4, \dots \} \\
 x_4 &\longleftrightarrow \{ x_2, \dots \} \\
 x_5 &\longleftrightarrow \{ x_0, x_4, x_5, \dots \} \\
 \dots &\longleftrightarrow \{ \dots \}
 \end{aligned}$$

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$		...
x_1 ↔				$x_3,$		$x_5,$	...
x_2 ↔	$x_0,$		$x_2,$			$x_5,$	...
x_3 ↔		$x_1,$			$x_4,$		...
x_4 ↔			$x_2,$				...
x_5 ↔	$x_0,$				$x_4,$	$x_5,$	...
... ↔	...	...	...	...	...	...	...

{ $x_0,$ $x_2,$ $x_5,$... }

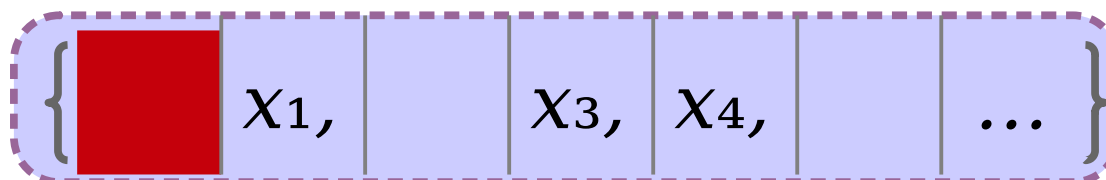
Which element is
paired with this
set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$		...
x_1 ↔				$x_3,$		$x_5,$	...
x_2 ↔	$x_0,$		$x_2,$			$x_5,$	...
x_3 ↔		$x_1,$			$x_4,$		...
x_4 ↔			$x_2,$				...
x_5 ↔	$x_0,$				$x_4,$	$x_5,$	...
...	...	...	...	...	...	...	...

"Flip" this set.
Swap what's
included and
what's excluded.

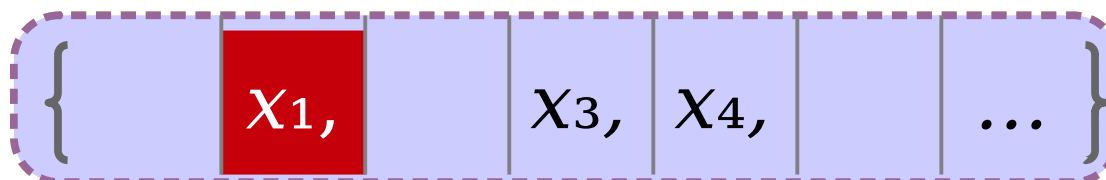
{ $x_1,$ $x_3,$ $x_4,$... }

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	x_0		$x_2,$		$x_4,$		...
x_1				$x_3,$		$x_5,$	...
x_2	$x_0,$		$x_2,$			$x_5,$	...
x_3		$x_1,$			$x_4,$		...
x_4			$x_2,$				...
x_5	$x_0,$				$x_4,$	$x_5,$	...
...	...	...	...	...	...	...	...



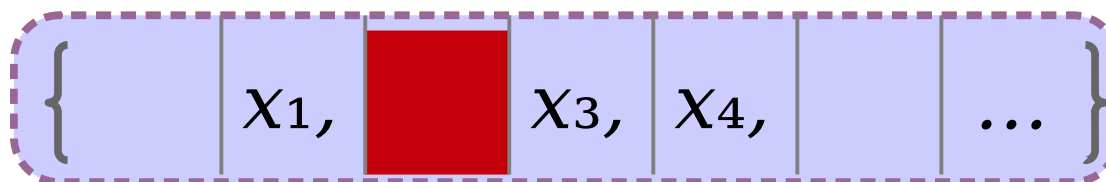
Which element is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$		...
x_1 ↔				$x_3,$		$x_5,$	...
x_2 ↔	$x_0,$		$x_2,$			$x_5,$	...
x_3 ↔		$x_1,$			$x_4,$		...
x_4 ↔			$x_2,$				...
x_5 ↔	$x_0,$				$x_4,$	$x_5,$	...
... ↔	...	...	...	...	...	...	...

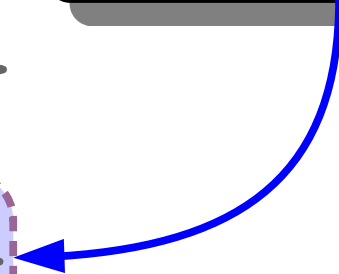


Which element is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$		...
x_1 ↔				$x_3,$		$x_5,$	...
x_2 ↔	$x_0,$		x_2			$x_5,$	...
x_3 ↔		$x_1,$			$x_4,$		...
x_4 ↔			$x_2,$				...
x_5 ↔	$x_0,$				$x_4,$	$x_5,$	...
...	...	...	...	...	...	...	...



Which element is paired with this set?



	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$		...
x_1 ↔				$x_3,$		$x_5,$	...
x_2 ↔	$x_0,$		$x_2,$			$x_5,$	...
x_3 ↔		$x_1,$			$x_4,$		...
x_4 ↔			$x_2,$				...
x_5 ↔	$x_0,$				$x_4,$	$x_5,$	...
...	...	...	...	...	...	...	...

{		$x_1,$		$x_3,$	$x_4,$		...}
---	--	--------	--	--------	--------	--	------

Which element is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
$x_0 \leftrightarrow$	$x_0,$		$x_2,$		$x_4,$		...
$x_1 \leftrightarrow$				$x_3,$		$x_5,$	...
$x_2 \leftrightarrow$	$x_0,$		$x_2,$			$x_5,$	...
$x_3 \leftrightarrow$		$x_1,$			$x_4,$		...
$x_4 \leftrightarrow$			$x_2,$				...
$x_5 \leftrightarrow$	$x_0,$				$x_4,$	$x_5,$	...
...	...	...	...	...	...	...	...

{ $x_1,$ $x_3,$ $x_4,$... }

Which element is paired with this set?

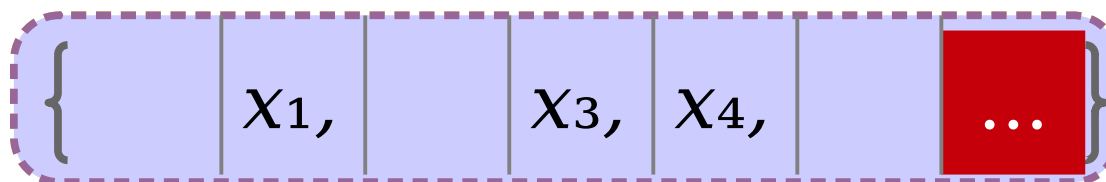
	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$		...
x_1 ↔				$x_3,$		$x_5,$	...
x_2 ↔	$x_0,$		$x_2,$			$x_5,$	...
x_3 ↔		$x_1,$			$x_4,$		...
x_4 ↔			$x_2,$				...
x_5 ↔	$x_0,$				$x_4,$	x_5	...
...	...	...	...	...	...	...	...

{ $x_1,$ $x_3,$ $x_4,$... }

Which element is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0 ↔	$x_0,$		$x_2,$		$x_4,$		...
x_1 ↔				$x_3,$		$x_5,$	...
x_2 ↔	$x_0,$		$x_2,$			$x_5,$	...
x_3 ↔		$x_1,$			$x_4,$		...
x_4 ↔			$x_2,$				...
x_5 ↔	$x_0,$				$x_4,$	$x_5,$	...
... ↔	...	...	...	...	...	...	...

Which element is
paired with this
set?



Two Infinities...

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

... and Beyond!

- By Cantor's Theorem:

$$|\mathbb{N}| < |\wp(\mathbb{N})|$$

$$|\wp(\mathbb{N})| < |\wp(\wp(\mathbb{N}))|$$

$$|\wp(\wp(\mathbb{N}))| < |\wp(\wp(\wp(\mathbb{N})))|$$

$$|\wp(\wp(\wp(\mathbb{N})))| < |\wp(\wp(\wp(\wp(\mathbb{N}))))|$$

...

- ***Not all infinite sets have the same size!***
- ***There is no biggest infinity!***
- ***There are infinitely many infinities!***

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
- 8. Cantor's Theorem**
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
9. Shocking Implications

Introduction to Set Theory

1. Sets and Elements (Definitions and Notation)
2. Subsets
3. The Empty Set
4. Set Builder Notation and Some Important Sets
5. Set Operations
6. Power Sets
7. Cardinality
8. Cantor's Theorem
- 9. Shocking Implications**

Implications

How does this have any practical consequences?

What does this have to do with computation?

Let's examine two sets:

- 1. "The set of all computer programs"*
- 2. "The set of all problems to solve"*

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings (*see appendix!*).

$$|\mathbf{Programs}| \leq |\mathbf{Strings}| < |\wp(\mathbf{Strings})| \leq |\mathbf{Problems}|$$

Every computer program is a string.

So, the number of programs is at most the number of strings.

From Cantor's Theorem, we know that there are more sets of strings than strings.

There are at least as many problems as there are sets of strings (*see appendix!*).

|Programs| < |Problems|

***There are more problems to
solve than there are programs
to solve them!***

|Programs| < |Problems|

It Gets Worse

- Using more advanced set theory, we can show that there are *infinitely more* problems than solutions.
- In fact, if you pick a totally random problem, the probability that you can solve it is *zero*.
- ***More troubling fact:*** We've just shown that *some* problems are impossible to solve with computers, but we don't know *which* problems those are!
- ***Key goal:*** We need to develop a more nuanced understanding of computation.

Where We're Going

- ***What makes a problem impossible to solve with computers?***
 - Is there a deep reason why certain problems can't be solved with computers, or is it completely arbitrary?
 - How do you know when you're looking at an impossible problem?
 - Are these real-world problems, or are they highly contrived?

Where We're Going

- ***How do we know that we're right?***
 - How can we back up our pictures with rigorous proofs?
 - How do we build a mathematical framework for studying computation?

Next Time

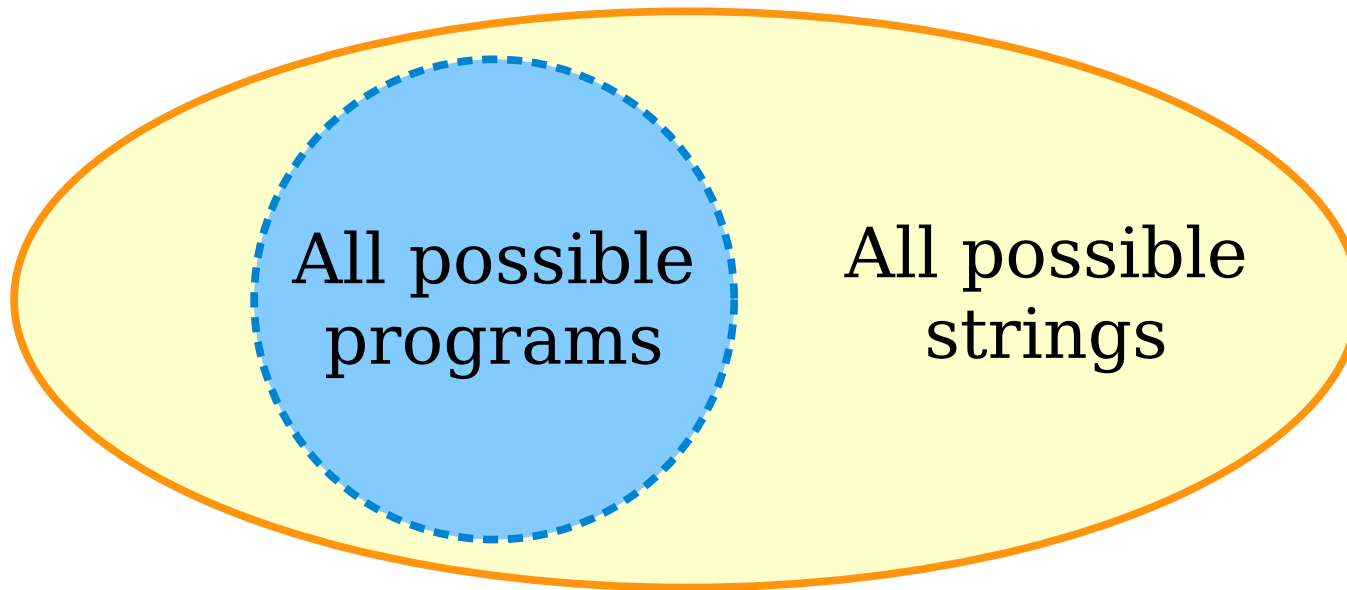
- ***Mathematical Proof***
 - What is a mathematical proof?
 - How can we prove things with certainty?

See also: Appendix: Strings and Problems

Appendix: Strings and Problems

Strings and Programs

- The source code of a computer program is just a (long, structured, well-commented) string of text.
- All programs are strings, but not all strings are necessarily programs.



$$|\mathbf{Programs}| \leq |\mathbf{Strings}|$$

Strings and Problems

- There is a connection between the number of sets of strings and the number of problems to solve.
- Let S be any set of strings. This set S gives rise to a problem to solve:

Given a string w , determine whether $w \in S$.

Strings and Problems

Given a string w , determine whether $w \in S$.

- Suppose that S is the set

$$S = \{ "a", "b", "c", \dots, "z" \}$$

- From this set S , we get this problem:

Given a string w , determine whether w is a single lower-case English letter.

Strings and Problems

Given a string w , determine whether $w \in S$.

- Suppose that S is the set
 - $S = \{ "0", "1", "2", \dots, "9", "10", "11", \dots \}$
- From this set S , we get this problem:

Given a string w , determine whether w represents a natural number.

Strings and Problems

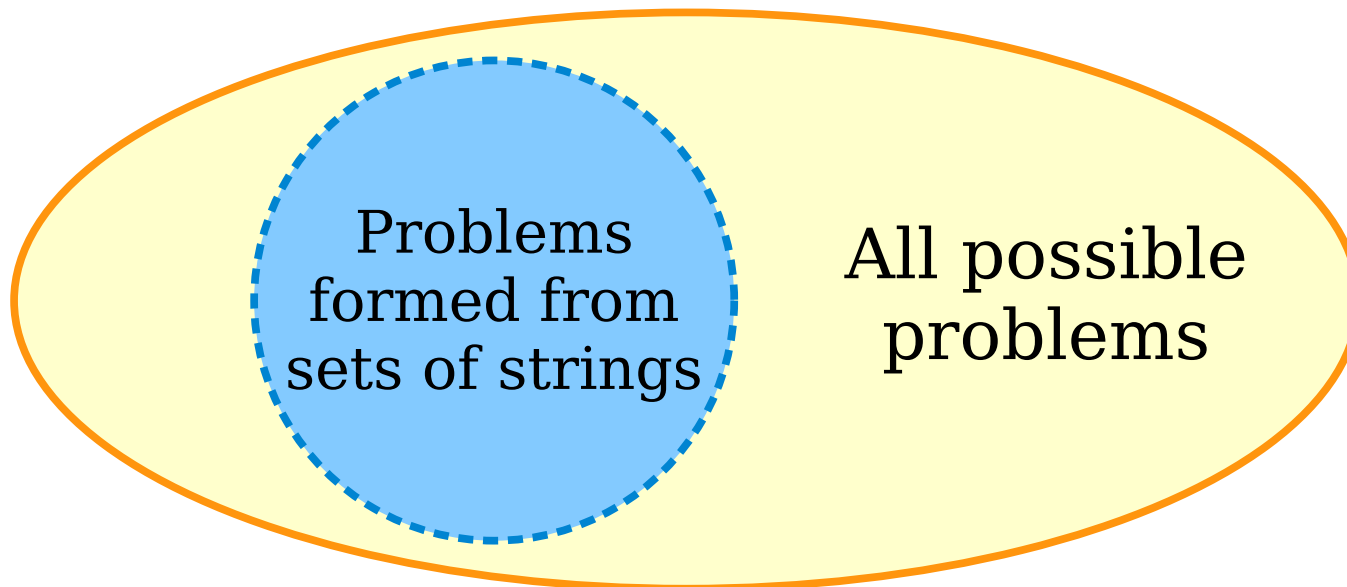
Given a string w , determine whether $w \in S$.

- Suppose that S is the set
 - $S = \{ p \mid p \text{ is a legal C++ program} \}$
- From this set S , we get this problem:

Given a string w , determine whether w is a legal C++ program.

Strings and Problems

- Every set of strings gives rise to a unique problem to solve.
- Other problems exist as well.



$$|\wp(\mathbf{Strings})| \leq |\mathbf{Problems}|$$